KOLAM DESIGNS BASED ON FIBONACCI NUMBERS

Part I: Square and Rectangular Designs.

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Kolams are decorative designs in South Indian folk-art. They are drawn around a grid of dots. In Part I of this two-part article, we describe a general scheme for *kolam* designs based on numbers of the Fibonacci series (0, 1, 1, 2, 3, 5, 8, 13, 21, 34 ....). Square *kolams* (3^2, 5^2, 8^2, 13^2, 21^2) and rectangular *kolams* (2 x 3, 3 x 5, 5 x 8, 8 x 13) are presented. The modular approach permits extension to larger *kolams* and computer-aided design. This enhances the level of creativity of the art. In Part II, we generalize the algorithmic procedure for *kolam* design to *kolams* of arbitrary size using Generalized Fibonacci numbers. Enumeration of *kolams* of small size (3^2, 5^2, 2 x 3) is described. Possible connections between *kolams* and knots are mentioned.

1. Introduction.

Kolams are decorative geometrical patterns that adorn the entrances of households and places of worship especially in South India. *Kolam* is a line drawing of curves and loops around a regular grid of points. Usually *kolams* have some symmetry (e.g. four-fold rotational symmetry). There are variants: (1) lines without dots (2) lines connecting dots and (3) free geometric shapes without lines or dots. The last variety has sometimes brilliant colours and is known as *Rangoli,* popular in North India. The traditional South Indian *kolam*, based on a grid of points is known as *PuLLi Kolam* or *NeLi Kolam* in Tamil Nadu (*PuLLi* = dot, *NeLi* = curve); *Muggulu* in Andhra Pradesh, *Rangavalli* in Karnataka and *Pookalam* in Kerala. Special *kolams* are drawn on festive occasions with themes based on seasons, nature (flowers, trees) religious topics and deities. Large *kolams* with bewildering complexity are common on such occasions. The folk-art is handed down through generations of women from historic times, dating perhaps thousand years or more. Constrained only by some very broad rules, *kolam* designs offer scope for intricacy, complexity and creativity of high order, nurtured by the practitioners, mostly housemaids and housewives, both in rural and urban areas. Here we deal only with *PuLLi Kolam* or *kolam* for short. For more information on *kolam* designs see Wikipedia [1] and the website [2].

In this article we describe a class of *kolams* based on the well-known Fibonacci numbers. Fibonacci numbers are

$$0 \ \ 1 \ \ 1 \ \ 2 \ \ 3 \ \ 5 \ \ 8 \ \ 13 \ \ 21 \ \ 34 \ \ 55 \ \ 89 \ \ 144 \ldots\ldots\ldots \qquad (*)$$

They belong to the infinite Fibonacci series generated by the following simple rule: given the first two numbers in the series 0 and 1, every other number is the sum of the preceding two (e.g. $1 = 0 + 1$, $3 = 2 + 1$, $5 = 3+2$ ....$144 = 89 +55$...). Fibonacci numbers are pervasive in all areas of science. In geometry they are related to pentagons, decagons and 3-dimensional Platonic solids. Rectangles of consecutive Fibonacci numbers as sides (e.g. 2 x 3, 3 x 5, 5 x 8, 8 x 13 ..) are known as Golden rectangles. The ratio of the sides approaches a limiting value denoted by $\varphi = (1+\sqrt{5})/2 = 1.61803....$ called the Golden Ratio. This ratio is ubiquitous in nature: in the branching of trees, arrangement of flower petals, seeds and leaves, spiral patterns of florets in sun flowers, spiral shapes of sea-shells etc. It appears to be closely related to growth processes in nature. The Golden Ratio is also believed to figure prominently in Western art: in architecture (pyramids, Parthenon), paintings (Leonardo da Vinci), sculpture, poetry (Virgil) and music. There is a vast literature on the subject [3] and many claims are controversial.

Fibonacci series first appeared in the book *Liber Abaci* (1202) by the Italian mathematician Leonardo of Pisa also known as Fibonacci. It is claimed that these numbers appeared in the analysis of prosody of Sanskrit poetry by Acharya Hemachandra in 1150 A.D, 50 years before *Liber Abaci* [4]. For an interesting account of Fibonacci numbers see Martin Gardner [5].

We will describe how *kolam* designs, square and rectangular, can be constructed using a modular approach. Bigger *kolams* are built from smaller ones using some properties of Fibonacci numbers. For example a square 'Fibonacci *kolam'* of 8^2 can be built using a square 2^2 and four rectangles of 3 x 5. Note that 8, 5, 3, 2 are all Fibonacci numbers. The modular approach is ideally suited for generating *kolams* on a computer and exploring interesting mathematical properties of *kolams* in general. In contrast to the controversial claims in Western art, here we *incorporate* Fibonacci numbers in a class of artistic designs we call Fibonacci *kolams* (FKs).

2. Kolam Designs.

A few simple common examples of *kolams* are shown in *Figure* 1. To facilitate our analysis of the mathematical properties of *kolams* based on Fibonacci numbers, we follow some ground rules.

Rule 1. *Uniformly spaced square and rectangular grids*. In *Figure* 1 there are grids of different types. We consider only grids like (a) to (f) and (k), but not grids like (g) to (j).

Rule 2. *Four-fold symmetry for square kolams*. When the design is rotated by 90^o, 180^o, 270^o about an axis perpendicular to the paper, the design remains the same.

Rule 3. *No empty unit cells*. Each dot in a square or rectangular grid may be considered as the centre of a four-sided unit cell. This rule requires that every unit cell in the *kolam* has a dot in the centre.

Rule 4. *Single loop*. The line traversing all the dots is a single endless loop.

To see how the rules work, we examine the *kolams* in *Figure* 1. (g) to (j) have no square or rectangular grid and are excluded (rule 1). (b) and (i) have empty unit cells violating rule 3. (c) a 3^2 contains three loops (two crescents and one circle) and has no four-fold symmetry. (f) a 4^2 has four-fold symmetry but contains two loops. The only *kolams* that meet all our criteria are (a),(d),(e),(k).

As the art is practised today, four-fold symmetry for square *kolams* is considered mandatory. (Hereafter, in the text by 'symmetry' we imply four-fold rotational symmetry). Grids other than square and rectangular, like the diamond-shaped ones in *Figures* 1(g), 1(h), 1(i) and the grid with extra rows of dots as in

Figure 1(j) are very popular [6]. Further, empty unit cells and multiple loops are generally allowed. Such departures from our rules, do not necessarily detract from the aesthetic value of *kolams*. Loose ends are forbidden. In a quaint, imaginative interpretation of the rule in folklore, closed loops capture evil spirits and prevent them from entering the home.

3. Fibonacci Numbers

As stated already, the Fibonacci numbers F_n ($n = 0, 1, 2...$) are defined by

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad (n > 1) \tag{1}$$

giving the Fibonacci series

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \quad 144 \ \ldots\ldots\ldots\ldots \tag{*}$$

Consider a set of any four consecutive Fibonacci numbers : Q = $(a \ \ b \ \ c \ \ d)$, e.g. (2 3 5 8) or (3 5 8 13). a, b, c, d are related as follows:

$$bc = b^2 + ab \tag{2a}$$

$$d^2 = a^2 + 4bc \tag{2b}$$

Proof: From equation (1) $c = b + a$. Multiplying both sides by b gives equation (2a). Again from equation (1)

$$d - a = b + c - a = b + b = 2b$$

$$d + a = c + b + a = c + c = 2c$$

Multiplying both sides of the two equations

$$d^2 - a^2 = 2b.2c = 4bc$$

which is the same as equation (2b).

A rectangle $b \times c$ is made up of a square b^2 and another rectangle $a \times b$ (equation 2a). A square d^2 is composed of a smaller square a^2 and four rectangles $b \times c$ (equation 2b). These are illustrated in *Figures* 2(a), 2(b). Note that the rectangle in *Figure* 2(a) is rotated by 0°, 90°, 180°, 270° in a cyclic order to make up the four rectangles ($b \times c$) in *Figure* 2(b), leaving a square a^2 in the centre. To make the connection to *kolams* clear, in *Figure 2*(b) the rectangles are 3 x 5 grids of dots and the central square is a 2^2 grid fitting in a larger square of 8^2 dots. But equations (2a) and (2b) are true for any quartet Q = *(a b c d)*. The reader would have noticed that the lengths *a, b, c , d* are all measured as "number of dots" in a line and not as lengths of the lines.

The most striking feature of a square FK is the *built-in* four-fold rotational symmetry of the square d^2 required by rule 2. In *Figure* 2(b), whatever be the symmetry (or lack of it) of the basic rectangle (*b x c*), its placement in four cyclic patterns ensures the overall four-fold symmetry of the big square (d^2). It is assumed that the central square a^2 has four-fold symmetry. To illustrate this symmetry note that the 'FK' symbol pattern in each rectangle has no symmetry and yet the overall symmetry of the big square is preserved. For example starting with Q = (2 3 5 8), one can assemble an 8^2 with a 2^2 at the centre surrounded by four rectangles of 3 x 5. (The 3 x 5 rectangle is in turn earlier built as a composition of a 3^2 and a 2 x 3 rectangle). This iterative process helps to build square and rectangular FKs of any size. Expressing equations (2a) and (2b) in standard notation

$$F_{n-2}F_{n-1} = F_{n-2}^2 + F_{n-2}F_{n-3} \qquad (n > 2) \tag{3a}$$

$$F_n^2 = F_{n-3}^2 + 4F_{n-2}F_{n-3} \qquad (n > 3) \tag{3b}$$

In the extensive literature on Fibonacci numbers, equation (3b) is either not mentioned or rarely mentioned. Equations (3a) and (3b) are the key tools for constructing a class of artistic symmetric square *kolams.*

4. 5^2 Fibonacci *Kolam.*

Square 5^2 FKs are based on Q = (1 2 3 5) with $5^2 = 1^2 + 4$ (2x3). In *Figure* 3(a), the 5^2 has four 2 x 3 rectangles (*Figure* 1a). There is a single dot (1^2) at the centre. The rectangles are merged with the central dot in a four-way splice shown by dotted lines. The resulting single loop *kolam* is shown in *Figure* 3(b). I saw *kolams* of this kind first in a 32-page supplement issued with a Tamil women's magazine [7]. Seven 2 x 3 *kolams* were named after the musical notes *Sa Ri Ga Ma Pa Da Ni*. The 2 x 3 *kolam* in *Figure* 3(a) is called *Sa*. Any other 'note' can be used in *Figure* 3(a). The splicing techniques to merge the four rectangles with each other and with the central dot, were further elaborated in another supplement of a Tamil magazine [8]. While the four-way splice at the centre (indicated by ◊) in *Figure* 3(b) always yields a single loop, other splicing points yield even more intricate designs. It turns out that the choice of splicing points determines the number of loops; only a restricted splicing pattern will yield a single loop. The *kolams* presented in the two booklets [7,8] have symmetry, but permit empty unit cells and multiple loops. However, single loop *kolams* are special and are called *anthadhi kolams* (*antham* = end, *adhi* = beginning) [7]. As we see later, rule 2 is easily satisfied, but it requires more careful work to satisfy rules 3 and 4. The 2 x 3 *kolam* in *Figure* 1(b) is the note *Ni* which is excluded in our analysis because of empty units cells.

In *Figure* 3(c), a copy of *Figure* 3(a), additional splicing points are marked by the symbols X and +. They are in addition to the four-way splice at the centre (◊). Note there are four symmetrically located points for X and for +. This is required to preserve the overall symmetry. The resulting *kolam* is *Figure* 3(d). Here the splicing points are indicated by dark dots. To illustrate the effect of the choice of splicing points on the number of loops, one of the three sets of splices (X + ◊) is omitted in *Figures* 3(e), 3(f), 3(g). When + is omitted we have five loops, four circles and one other loop (*Figure* 3e). When X is omitted there are three loops (*Figure* 3f) and when ◊ is omitted there are again five loops, one of them being the loop around the central dot (*Figure* 3g). Single loop is obtained only for all the three splices (X + ◊) or just ◊. (*Figures* 3b and 3d). Multiple loops are drawn in different colours.

The number of possible 5^2 FKs depends on the number of distinct 2 x 3 rectangles and the number of splicing choices. There are 30 possible distinct 2 x 3 rectangles each giving a different 5^2 *kolam*. We will return later (Part II) to this enumeration problem.

5. 3 x 5 Rectangular Fibonacci *Kolam*.

A 3 x 5 FK is obtained by splicing together a 3^2 and 2 x 3 rectangle (3 x 5 = 3 x 3 + 2 x 3). Here the number of choices increases dramatically because there is no requirement of four-fold symmetry; further the single-loop condition applies only to the *whole* FK (3 x 5) and not to the constituents (3^2 or 2 x 3). By a proper choice of splicing points, multiple loops can all merge into a single loop. For example in *Figure* 4(a) are shown the 2 x 3 rectangle on the left (*Figure 1*a) and the 3^2 with three loops on the right (*Figure* 1c). The two are spliced together at three points marked X and the result is a single loop 3 x 5 FK (*Figure* 4b). This happens to be the same as *Figure* 1(k), a very popular commonplace *kolam*.

We will require a symmetric 3^2 FK when we assemble a 13^2 FK based on the quartet Q = (3 5 8 13): $13^2 = 3^2 + 4\ (5 \times 8)$. The central 3^2 has to be symmetric. There is only one symmetric single loop 3^2 and this is shown in *Figure* 4(d). This in turn is obtained as an FK from Q = (1 1 2 3): $3^2 = 1^2 + 4\ (1 \times 2)$. The constituents are four 1 x 2 rectangles and a central dot (1^2) shown in *Figure* 4(c). The four-way splice at the centre gives *Figure* 4(d). The enumeration of all possible 3 x 5 FK rectangles is a complex combinatorial problem. We will discuss it briefly later.

6. 8^2 Fibonacci *Kolam*.

With the modules of 3 x 5 rectangles in hand (section 5) we can draw an 8^2 FK using Q = (2 3 5 8): $8^2 = 2^2 + 4\ (3 \times 5)$. We need a symmetric 2^2 at the centre. Surprisingly, there is some difficulty in drawing such a simple figure. The obvious first guess (*Figure* 5a) is not valid since it has an empty unit cell in the centre. We turn to the generator of 2 x 2 for a hint Q = (0 1 1 2): $2^2 = 0^2 + 4\ (1 \times 1)$. In *Figure* 5(b), the four 1 x 1 'rectangles' are the four circled dots and the 0^2 can be imagined as a small empty square at the centre with its area approaching 0, i.e. the square collapses to a point with dimension 0. The cloverleaf pattern with four leaves meeting at the centre is the obvious solution (*Figure* 5c). This pattern rarely occurs in conventional *kolams* because usually the line curves around the grid points. But the centre of the cloverleaf is not a grid point; it is an imaginary point and it occurs only at the centre of a square whose side is even (8 in this case).

We have chosen *Figure* 4(b) as the 3 x 5 FK rectangle. Four such rectangles (in four colours) fill the 8^2 in *Figure* 5(d) leaving a cloverleaf 2^2 at the centre. The splicing locations are marked X, + and Δ located symmetrically on the four sides (12 splices). The result is the 8^2 single loop FK in *Figure* 5(e). There are other splicing choices yielding different FKs. One possible variation is to replace the cloverleaf at the centre by four isolated points (*Figure* 5b). These four points will technically be called four loops. It turns out that none of the simple splicing choices leads to a single loop; usually we get two loops.

7. Square and Rectangular Fibonacci *Kolams* of Higher Order.

The recursive relations (equations 3a and 3b) can be used to generate square and rectangular FKs of any order. This is illustrated in sections 4,5 and 6. The general procedure is graphically presented in *Figure* 6. There are three blocks (I, II, III) each with two columns. The middle block (II) contains the 'modules' used for building the 'square FKs' in the right block (III) and the 'rectangular FKs' in the left block (I). Modules are shown in italics in block II. For example, to find the composition of a 13^2 FK (in III) trace the arrows leading to the module II (5 x 8 and 3^2). Similarly the 8 x 13 rectangle in I, traces its origin to the module 8^2 and 5 x 8 in II. The origins of all the square and rectangular FKs can be traced ultimately to $F_1^2 = 1^2$ or $F_2^2 = 2^2$. Every third Fibonacci number F_{3m} (m = 1 2 3..) is even because in Q = *(a b c d)* , $d - a = c + b - a = 2c$ an even number. Therefore d and a are either both even or both odd. Since $F_3 = 2$ is even, so are F_6, F_9, F_{12} ... (8, 34, 144....). Since F_1=F_2= 1, all other Fibonacci numbers are odd.

In *Figure* 7 we present the rectangular FKs 5 x 8, 8 x 13 and the square FKs 13^2 and 21^2. In *Figure* 7(a), the 5^2 FK (*Figure* 3d) on the left and a 3 x 5 FK (*Figure* 4b) on the right are spliced together at the points shown as dark dots to produce a 5 x 8 FK. In *Figure* 7b, the 5 x 8 FK (*Figure* 7a) is on the left and the

8^2 FK on the right (*Figure* 5e). Using Q = (3 5 8 13): $13^2 = 3^2 + 4\ (5 \times 8)$, a 13^2 FK is built from four 5 x 8 rectangular FKs (*Figure* 7a) and a central 3^2 FK (*Figure* 4d). This is shown in *Figure* 7(c). The last FK, a 21^2 (*Figure* 7d) has four 8 x 13 rectangles (*Figure* 7b) and a central 5^2 (*Figure* 3d). It is based on Q = (5 8 13 21) : $21^2 = 5^2 + 4\ (8 \times 13)$. While the first three have each three sets of splices, the last one has six sets of splices (24 splices in all). They are indicated at the bottom of the 8 x 13 rectangle (*Figure* 7b) as X X X X Δ Δ. The first four Xs are splices between two rectangles and the last two are between a rectangle and the central square.

The recursive scheme to generate Fibonacci rectangles and squares (*Figure* 6) is algorithmic and can be used to draw FKs on a computer with suitable software. The user will have to experiment with different splicing patterns to obtain a single loop. If splicing points are adjacent, one often gets empty unit cells which are forbidden. One useful criterion to adopt is to aim for the maximum possible number of splicing points; the 21^2 FK in *Figure* 7(d) is an example. Further, we reiterate that for square FKs, splicing comes in sets of four symmetrically located points.

Computer-aided *kolam* design will elevate the art to a new level of creativity. A computer program becomes indispensable if one has to explore the different possible combinations of rectangular and square modules that constitute the *kolam* and the different choices of splicing points that yield single loops. It is also a mathematical challenge to enumerate all possible FKs of any given size. It appears feasible only for 2 x 3, 3^2 and 5^2 using manual methods. Although we have considered only square grids it should be possible to extend the results to other grids. For example the diamond-shaped *Figure* 1(h) has as its centre the 3^2 (*Figure* 1d) with four extra corner dots spliced to the four sides. Many examples of such extensions of *kolams* by adding extra modules on the four sides are given in [7,8]. A square FK can be converted into a diamond-shaped FK by adding four triangles on the four sides. This is elaborated in Part II.

8. Square and Rectangular *Kolams* of Arbitrary Size.

The Fibonacci *kolams* we have described are all based on the Fibonacci series (*). This is a particular example of a Generalized Fibonacci series G_n (n = 0 1 2 3......) where the first two 'starting' numbers G_o, G_1 are α, β. The Fibonacci series corresponds to the special case $\alpha = 0$ and $\beta = 1$. For example if $\alpha = 2$ and $\beta = 1$ we get the series

2 1 3 4 7 11 18 29 47 76 ....

known as the Lucas series. As long as the 'Fibonacci recursion'

$$G_n = G_{n-1} + G_{n-2} \qquad (n > 1) \tag{4}$$

is used the quartet relation Q = (a b c d) with equations (2a) and (2b) applies. In other words in equations (3a) and (3b), F_n can be replaced by G_n. We can exploit this fact to generate Generalized Fibonacci *Kolams* (GFK) of any order, i.e. n x n or m x n for all m and n. In particular 4 and all the succeeding numbers in the Lucas series are different from the Fibonacci numbers. For example a 4 x 4 GFK can be generated from Q = (2 1 3 4): $4^2 = 2^2 + 4\ (1 \times 3)$.

There exist some simple iterative schemes for *kolam* designs of arbitrary size which are special cases of Generalized Fibonacci numbers. All these aspects - *kolams* of arbitrary size, generating schemes, splicing patterns and their relation to the number of loops, enumeration and classification of *kolams* of a given size and extension to diamond-shaped grids - will be discussed in Part II of this article. It will also explore some possible interesting connections between *kolams,* Group theory and the mathematical theory of knots, a part of algebraic topology.

Suggested Reading.

[1] http://en.wikipedia.org/wiki/kolam

[2] www.ikolam.com

[3] Mario Livio. *Golden Ratio: Story of PHI the world's most outstanding number.* Broadway Books, 2003.

[4] Paramananda Singh. *Acharya Hemachandra and the (so called) Fibonacci numbers.* Math. Ed. Siwan Vol. 20, pp.28-30, 1986.

[5] Martin Gardner. *The Mathematical Circus*, The Mathematical Association of America, Washington D.C., 1992.

[6] *Figure* 1(j) is also the endless trajectory of a billiard ball in a rectangular billiards table. This is true for *Figure* 1(b) too. The ball is shot at an angle of 45^o. Martin Gardner in one of the articles in 1960's in his famous series *Mathematical Games* in *Scientific American,* describes '*Bouncing balls in polygons and polyhedrons'*. This author wrote to Gardner at that time about the similarity between some *kolams* and billiard ball trajectories. Gardner replied stating that he will comment on this if there arose another occasion to discuss mathematical billiards. The original article appears also in '*Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American'* (1971).

[7] Salem Meera. Supplement to Tamil magazine '*Kumudam Snehidi',* December 2005.

[8] Sumati Bharati. *Sarigamapadani Kolangal* Supplement to Tamil magazine '*Snehidi',* February 2007.

Figure Captions.

Figure 1. A sampling of some small popular *kolams*

Figure 2. Construction of a square Fibonacci *kolam*.

Figure 3. Construction of 4^2 Fibonacci *kolam*.

Figure 4. Fibonacci *kolams*: 3 x 5 rectangle and 3^2.

Figure 5. Fibonacci *kolams*: 2^2 and 8^2.

Figure 6. Recursive Procedure for Construction of Square and Rectangular Fibonacci kolams.

Figure 7. Fibonacci rectangular *kolams*: (a) 5 x 8 (b) 8 x 13. Fibonacci square *kolams* (c) 13^2 (d) 21^2.

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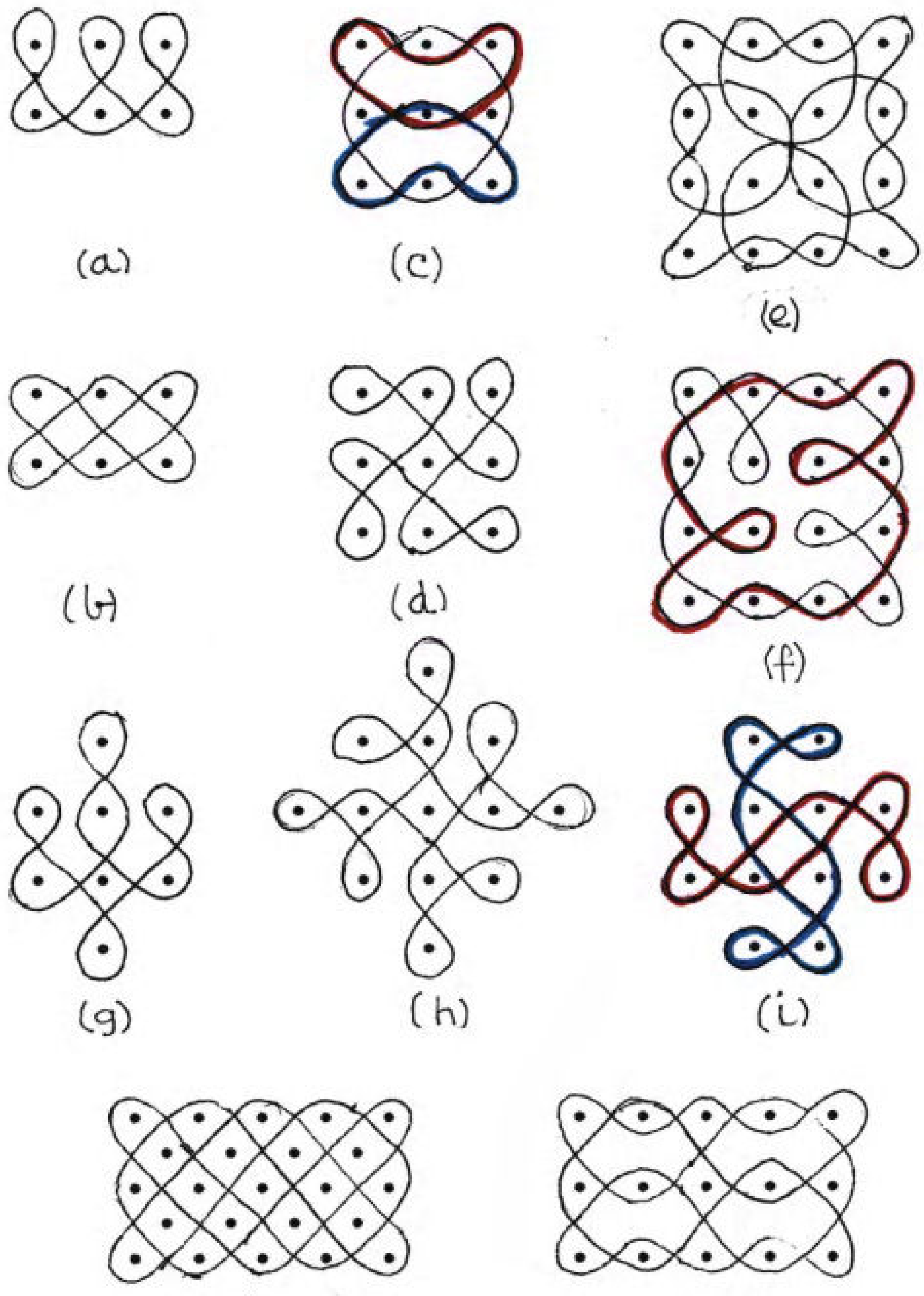

Figure 1

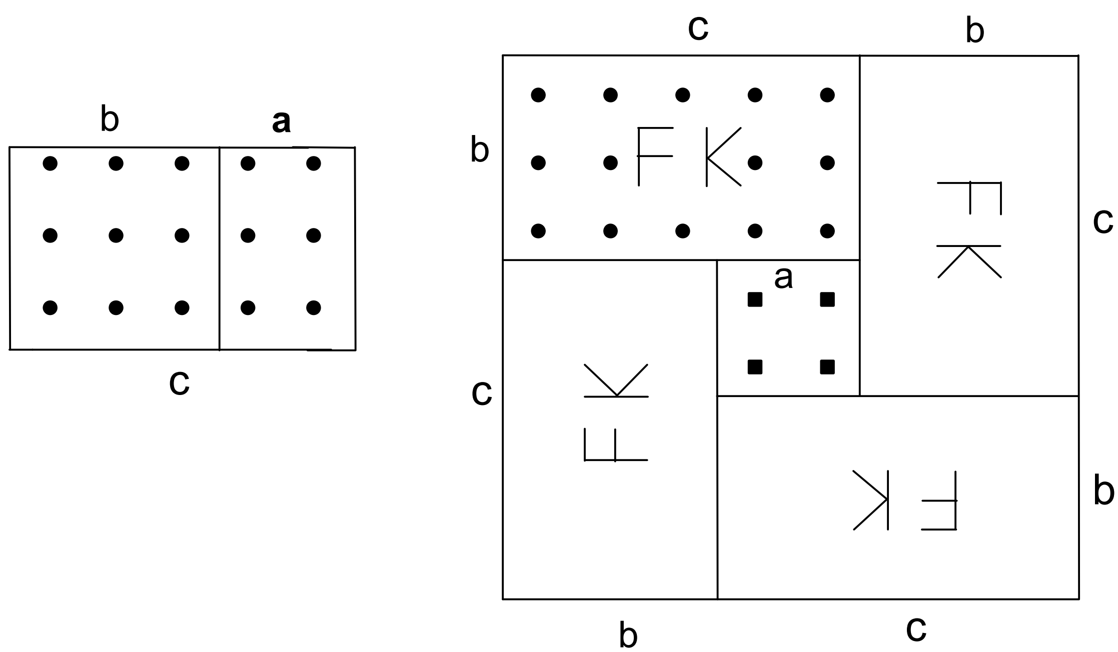

Figure 2

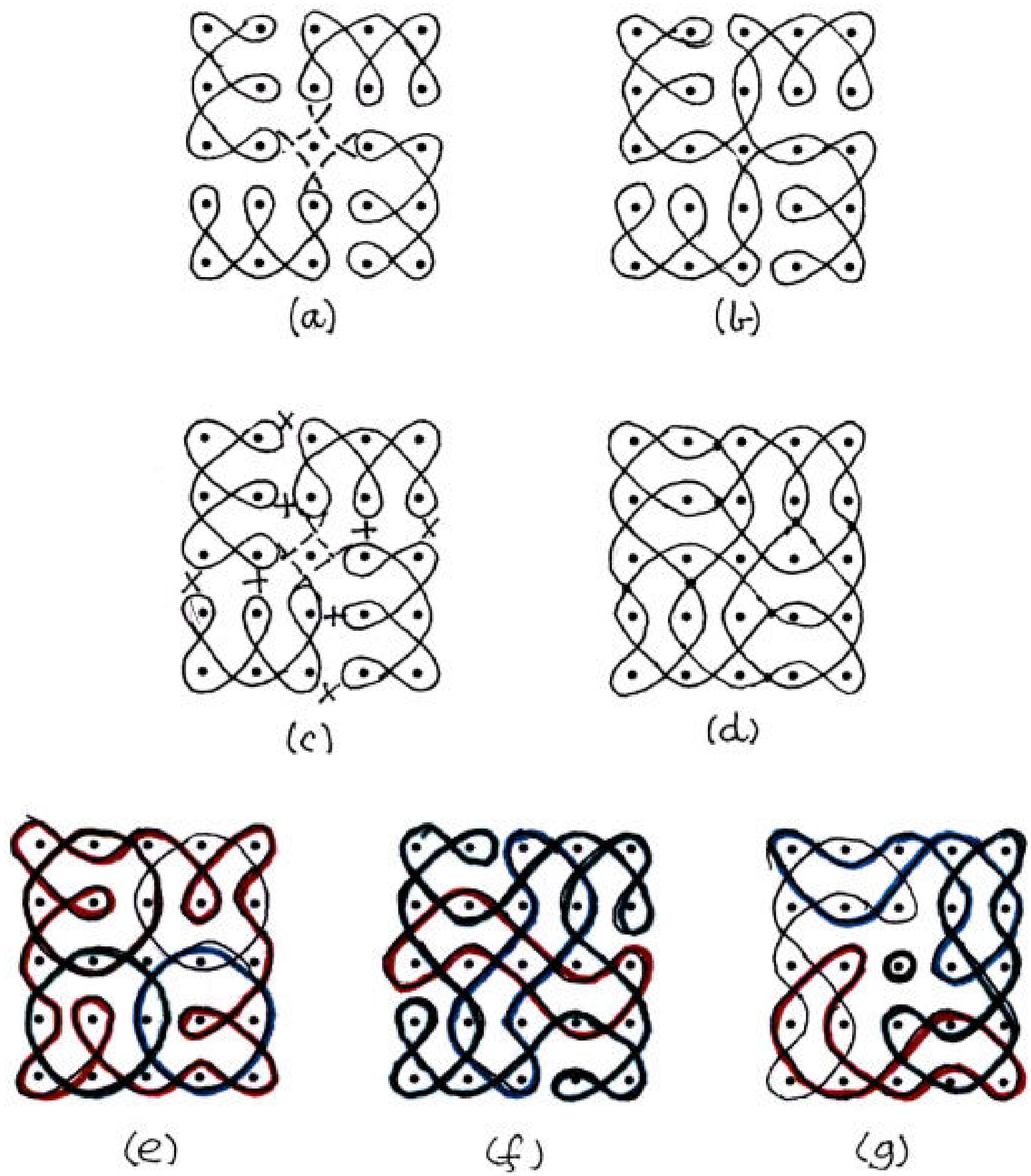

Figure 3.

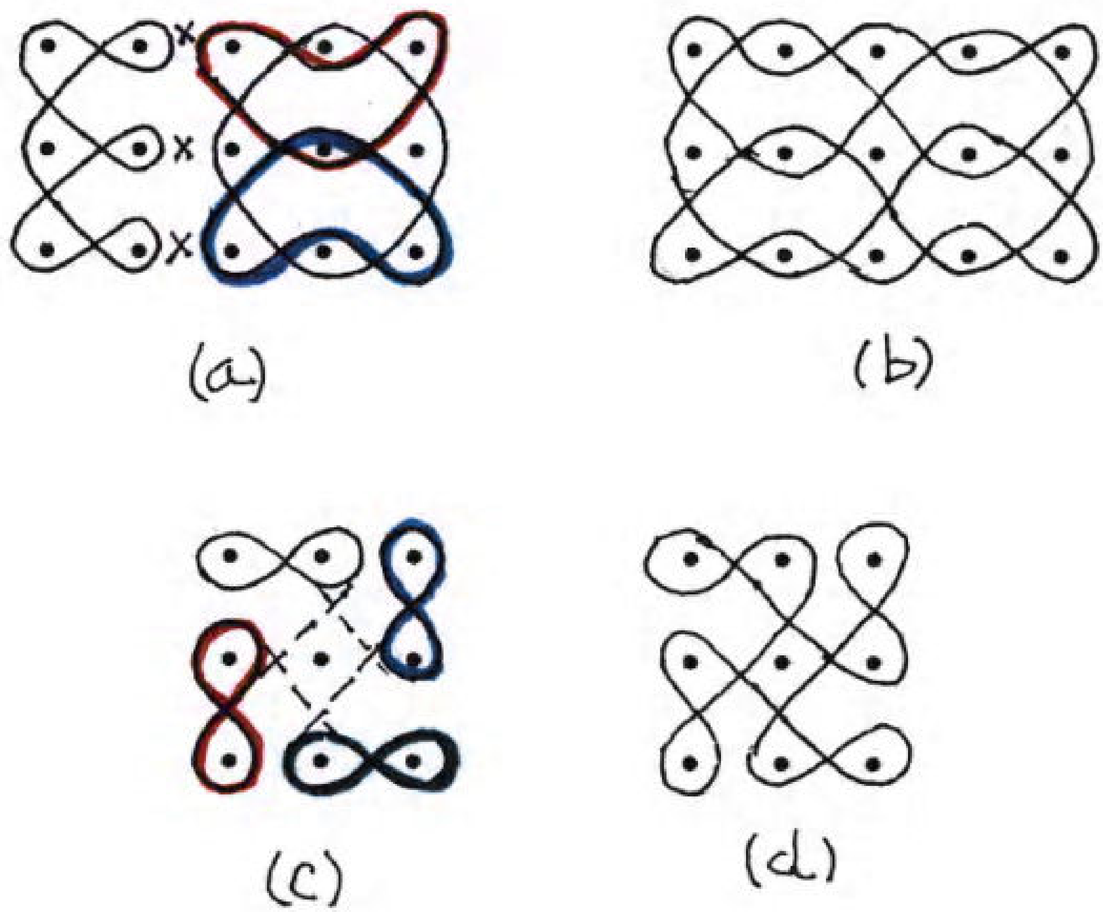

Figure 4

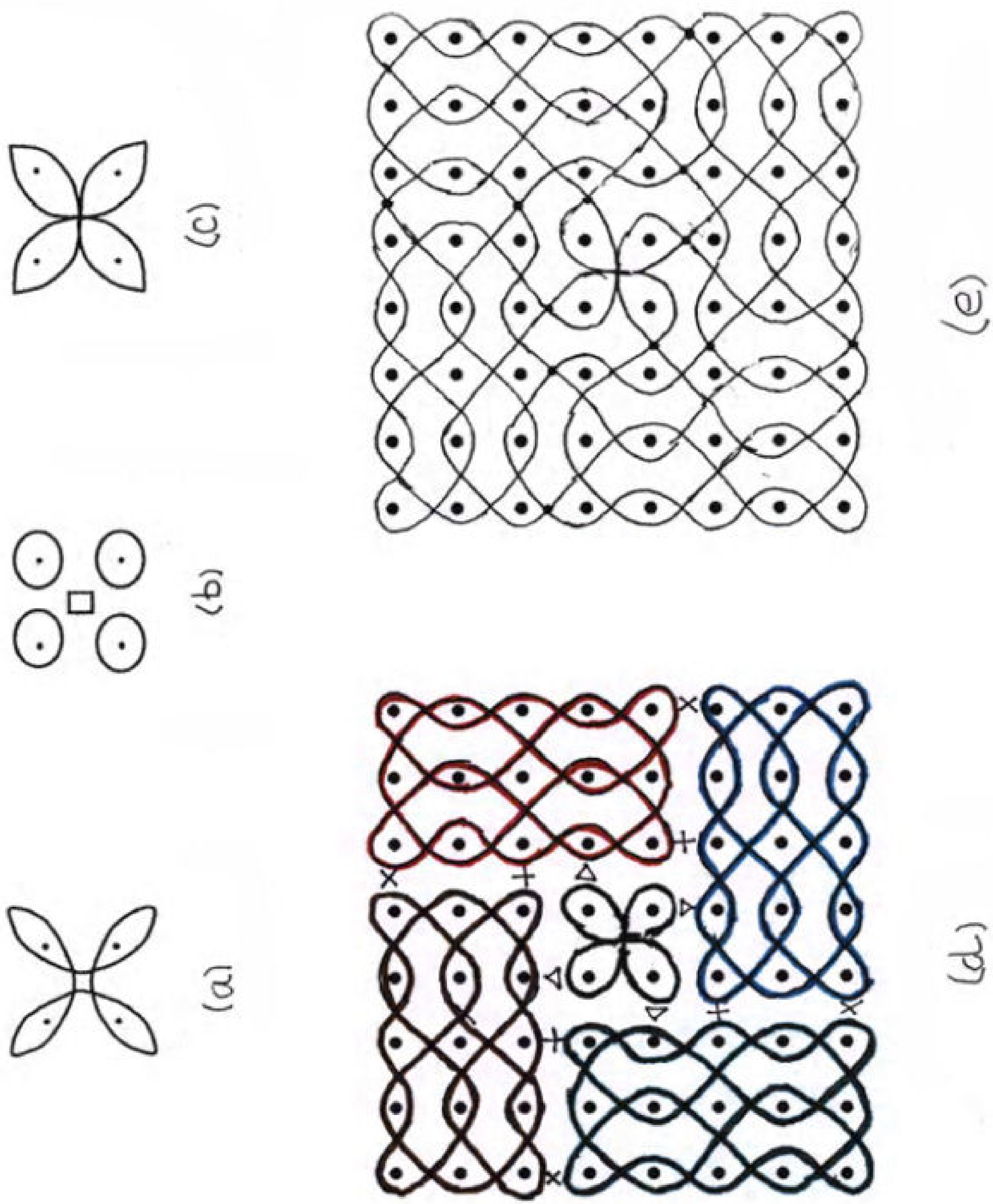

Figure 5

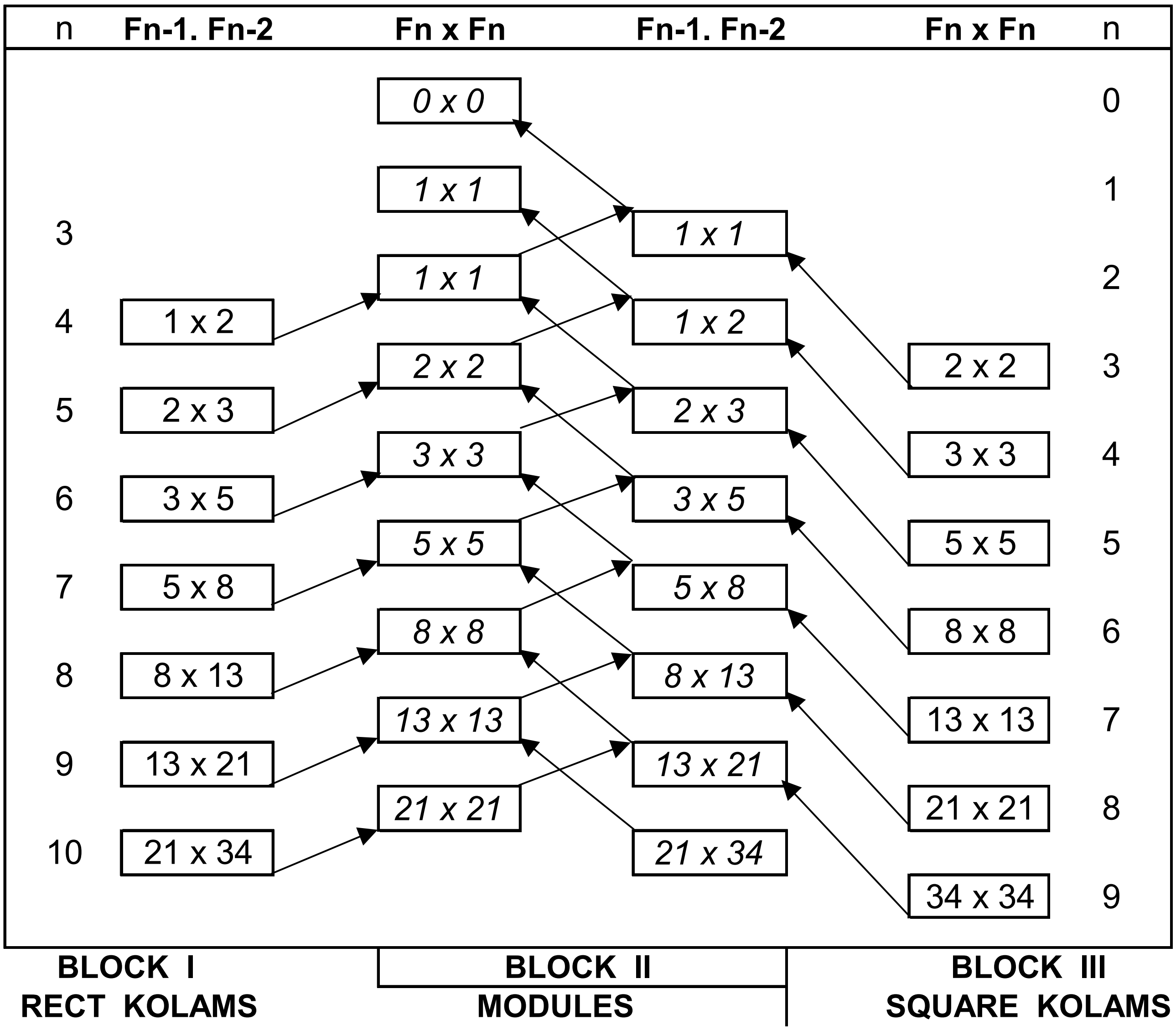

FIGURE 6. RECURSIVE PROCEDURE FOR CONSTRUCTION OF SQUARE AND RECTANGULAR FIBONACCI KOLAMS

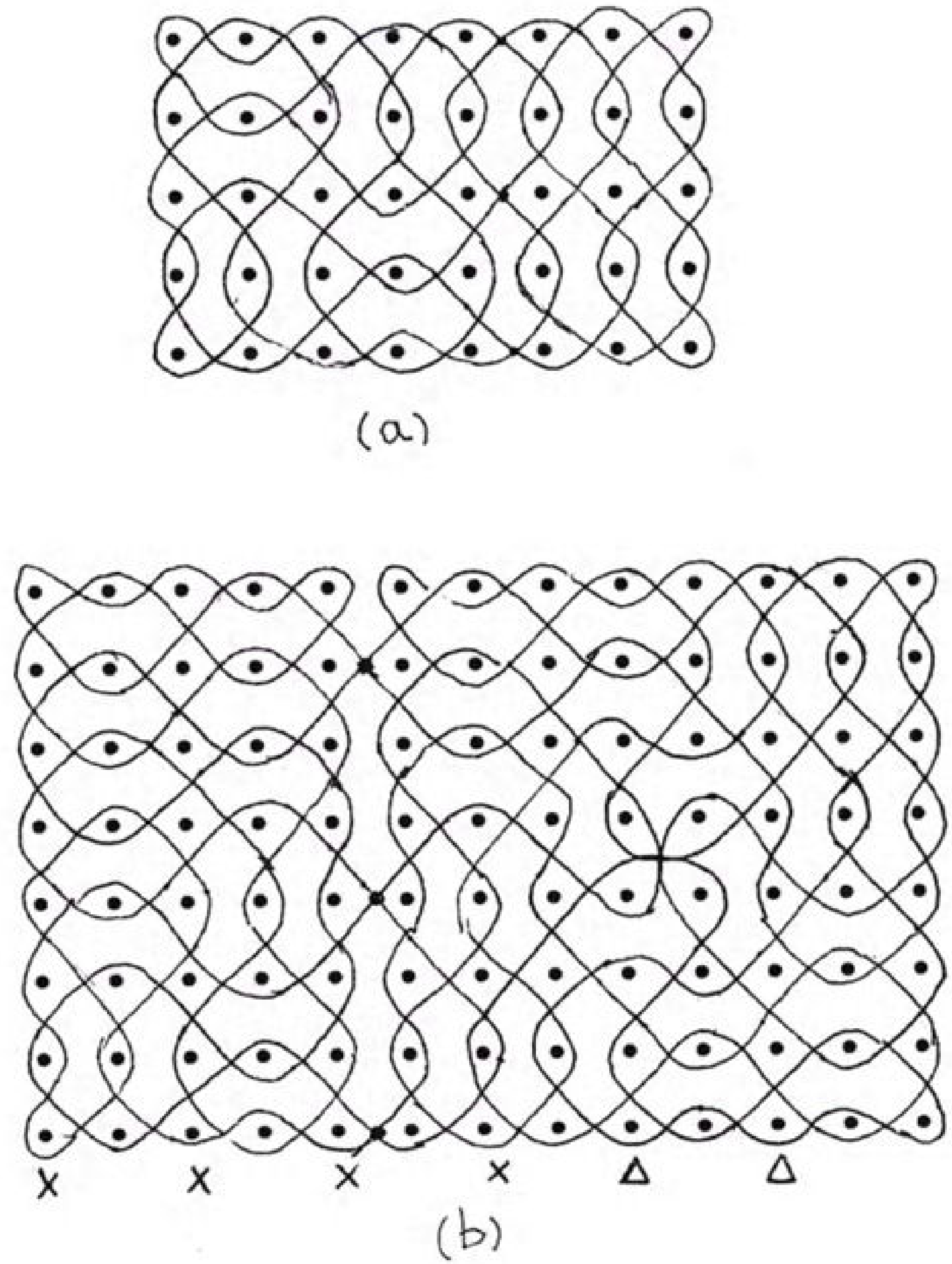

Figure 7

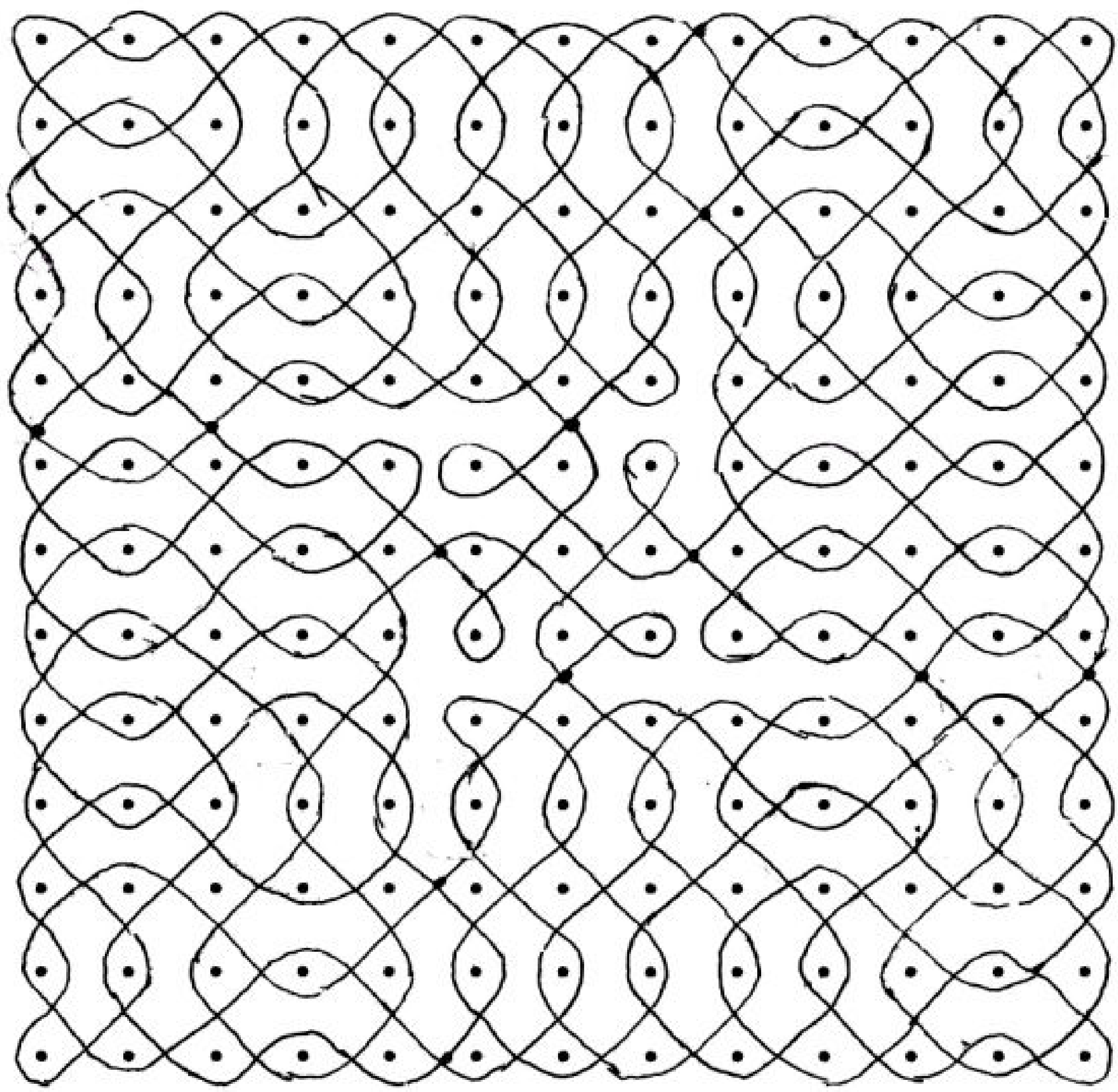

Figure 7(c)

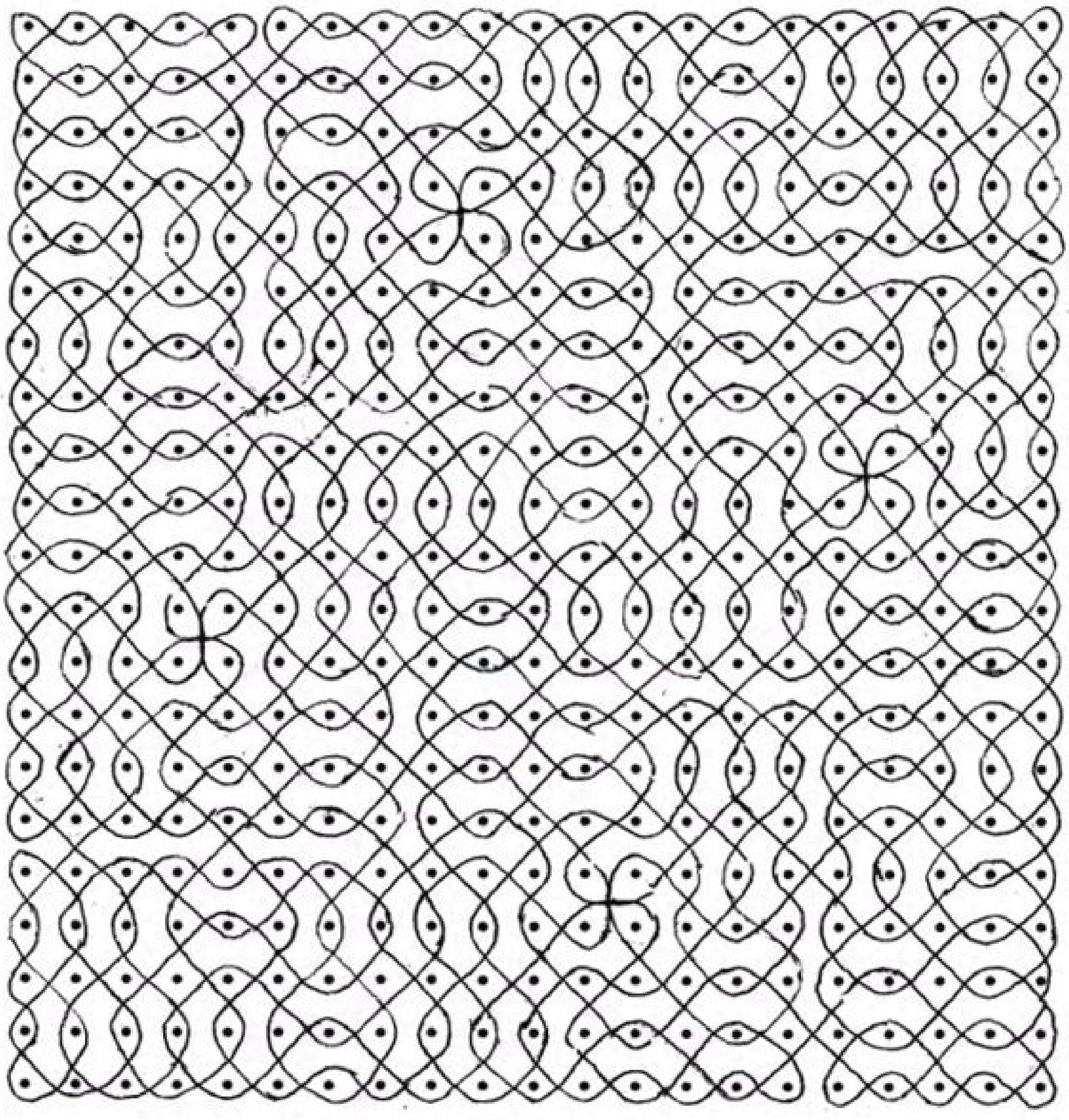

Figure 7(d)