

KOLAM DESIGNS BASED
ON FIBONACCI NUMBERS

Part IV. Evolution of Loops

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A B S T R A C T

A new family of Fibonacci *Kolams* based on Fibonacci Recurrence with four-fold rotational symmetry (square *kolams*) and two-fold rotational symmetry (rectangular *kolams*) was described in earlier papers in this series. A key feature is the modular scheme in which larger *kolams* are made from smaller ones by merging them at a set of splicing points. A cardinal feature of the *kolams* is that they are single-loop. The task is to splice the modules maximally, consistent with a single loop. Generally the outcome of a large number of splices is multiple loops. It is shown that as the number of splices increases (as in large *kolams*), the final outcome is a small set of odd-numbered loops (1 3 5). This follows from a set of empirical splicing rules that allows a formulation of loop evolution as binary trees; and invoking matrices of loop probabilities to determine the limiting probability matrix as the number of splices is increased.

‘Matrices are like the art of juggling balls’

‘What goes up must come down’

1. Introduction.

In this series of articles on Fibonacci *Kolams*, Parts I, II, III dealt with square and rectangular *kolams* based on Fibonacci Recurrence. Part I covered square *kolams* based on the canonical Fibonacci series: 0 1 1 2 3 5 8 13..... and Part II provided recipes for square *kolams* of arbitrary size with four-fold rotational symmetry [1][2]. This symmetry implies that the *kolam* appears the same viewed from all four sides North, East, South and West.

In a generalized Fibonacci series the first two numbers are free to choose and every subsequent number is sum of two preceding numbers, e.g.

3 4 7 11 18 29 47

(The first two numbers of the canonical Fibonacci series are 0 and 1). A square *kolam* is encoded by a quartet Q of consecutive Fibonacci numbers

$$Q(a \ b \ c \ d)$$

with $c = a + b$ and $d = c + b$. The four numbers are related.

$$d^2 = a^2 + 4bc \tag{1a}$$

A geometric interpretation of the above is: *a square of side d has a smaller square of side a concentric to it and four rectangles b x c cyclically placed to fill the space in between* (Figure 1). Whatever be the (symmetry of) rectangle $b \times c$, the figure looks the same viewed from four sides. This is of course true only if the small square a^2 has the same rotational symmetry.

The rectangles needed for the above construction can be generated using the relation

$$bc = b(b + a) = b^2 + ab \tag{1b}$$

Rectangle $b \times c$ is made up of a square b^2 and a rectangle $a \times b$. Equations (1a) and (1b) are the basis for building Fibonacci *Kolams* of ever increasing

size step by step starting from small *kolams*. The ratio of the sides of these rectangles approaches the Golden Ratio $\phi = ((1 + \sqrt{5})/2 = 1.61803 \dots$ as their size increases. Hence they are called Golden Rectangles.

Golden Rectangles have no symmetry (rotation or reflection). Rectangles can have only two-fold rotational symmetry (not four-fold) or reflection symmetry (about horizontal and/or vertical axis). However rectangular *kolams* with two-fold symmetry can be constructed using two Fibonacci quarters

$$\begin{aligned}
 & Q(a_1 \ b_1 \ c_1 \ d_1) \\
 & Q(a_2 \ b_2 \ c_2 \ d_2) \\
 & c_1 = a_1 + b_1 \quad d_1 = c_1 + b_1 \\
 & c_2 = a_2 + b_2 \quad d_2 = c_2 + b_2
 \end{aligned} \tag{2}$$

All the numbers are related.

$$d_1 d_2 = a_1 a_2 + 2 b_1 c_2 + 2 c_1 b_2 \tag{3}$$

Analogous to Equation (1a), the above has the following geometric interpretation: *a rectangle of area $d_1 d_2$ (height d_1 , width d_2) has a smaller rectangle $a_1 a_2$ at the centre; the space between them is filled with two pairs of rectangles $b_1 c_2$ and $c_1 b_2$.* (Figure 2). The figure appears the same viewed from North and South or from East and West provided the central rectangle has the same property [3].

A distinguishing feature of the Fibonacci *Kolams* is that they are single-loop. A Fibonacci *Kolam* is constructed by merging or splicing five smaller modules at a set of points. In general, the final *kolam* has multiple loops. By suitably choosing the splicing points, a single loop is achieved.

This article is devoted to the evolution of the number of loops as the number of splices is increased. It turns out that the evolution can be modelled in simple terms and mathematically studied. Surprisingly, it is found that the final number of loops approaches small limiting values (1 3 5) with well defined probabilities.

2.0. Evolution of Loops

In Figure 3, a 5^2 Fibonacci *Kolam* is drawn based on Q (1 2 3 5). Starting with five loops in (a), the final 5^2 emerges in (k) after a series of splices. At the centre of (a) is one dot (1^2) and four rectangles 3×2 are placed in a cyclic pattern.

The first set of splices is of the rectangles with the centre dot, called a “four-way splice” as shown in (b). All the five loops have merged into a single loop as in (c). One can stop here since a single loop is achieved. But there are available more splicing choices that can lead to more complex and intricate *kolams* of aesthetic value.

I illustrate the ‘evolution of loops’ as two more sets of splices PQRS and KLMN are added. Note that the splices come in sets of four to ensure four-fold rotational symmetry – e.g. a splice at P forces splices at Q R S as well. Same is true for the set KLMN. These splicing points are shown in (b). All the three sets of splices can be done in a single step taking (b) to (k). But to illustrate how the number of loops evolves, the splices are done one after another in (c) through (k).

Splice at P splits the single loop of (c) into two loops as in (d). Splice at Q merges the two into a single loop again as in (e). Splice at R gives two loops again as in (f) and splice at S splits again the loops into three as in (g). This sequence can be represented as

$$\begin{array}{ccccccccc}
 1 & \rightarrow & P & \rightarrow & 2 & \rightarrow & Q & \rightarrow & 1 & \rightarrow & R & \rightarrow & 2 & \rightarrow & S & \rightarrow & 3 \\
 (c) & & & & (d) & & & & (e) & & & & (f) & & & & (g)
 \end{array}$$

The next four Figures (h) (i) (j) (k) show the splices at KLMN respectively.

$$\begin{array}{ccccccccc}
 3 & \rightarrow & K & \rightarrow & 2 & \rightarrow & L & \rightarrow & 1 & \rightarrow & M & \rightarrow & 2 & \rightarrow & N & \rightarrow & 1 \\
 (g) & & & & (h) & & & & (i) & & & & (j) & & & & (k)
 \end{array}$$

a reversal of the sequence PQRS. The final result is a single loop in (k). Putting the two sequences together

$$(PQRS)(KLMN) \rightarrow (1-2-1-2-3)(3-2-1-2-1)$$

It is worth noting that the order in which the splices are done does not affect the final number of loops. So irrespective of the order of PQRSKLMN the final number of loops is one, although the evolutionary paths may be different [4]. For example

$$(KLMN)(PQRS) \rightarrow (1-2-3-4-5)(5-4-3-2-1)$$

2.1. Splicing and Unsplicing Rules

From a detailed analysis of loop evolutions the following empirical rules emerge.

- (a) a splice between two loops gives one loop
- (b) a splice within a single loop splits it into two loops

In practice a good strategy to obtain maximal splicing consistent with a single loop is the following. Splice all allowed splicing points at one ‘go’. If the number of loops is one, the task is done. If the number is more than one, then unsplice a set of four points to make the *kolam* single loop; the choice of this set will require some ‘trial and error’ experimentation.

As a corollary to the ‘splicing rules’, the ‘unsplicing rules’ are

- (a) unsplicing at the intersection of two loops give one loop
- (b) unsplicing within a single loop will split it into two loops.

The above rules are presented in Figure 4 as a 2 x 2 table. Splicing points are indicated by X and the unsplicing points by O. Where two loops appear, one of them is a dashed line. The splicing rules are shown as

$$C \rightarrow D \text{ (single loop)} \quad \text{and} \quad E \rightarrow F \text{ (two loops)}$$

The unsplicing rules are

$$F' \rightarrow E' \text{ (single loop)} \quad \text{and} \quad D' \rightarrow C' \text{ (two loops)}$$

Analysis of a large number of *kolams* shows that these rules are adequate.

However there is another possible situation which can occur in principle:

‘a splice within a single loop leaves the loop single’

The above is shown in Figure 4 as $A \rightarrow B$ and its converse as $B' \rightarrow A'$.

Both are shown boxed to indicate they occur rarely.

How do we account for this anomaly? In seeking an explanation, it was found that splicing points are of two types. In a single loop, the directions of traversal on either side of a splicing point are either parallel ($\uparrow\uparrow$) or anti-parallel ($\downarrow\uparrow$). On splicing, parallel splices result in two loops and anti-parallel ones leave the loop single. In Figure 4 anti-parallel case is illustrated as $A \rightarrow B$ and parallel case as $C \rightarrow D$. We take as a working hypothesis that *a splice in a single loop always splits the loop into two*. This leads to a simple model of the ‘evolution of loops’ as described in Section 3. Now we examine how the splicing points are chosen.

2.2. Choice of Splicing Points

In a square Fibonacci *Kolam* four rectangles are spliced to a central square. The merger occurs at points along the edges of rectangles. As already mentioned the choice of splices should ensure a single loop. Another restriction – self-imposed for aesthetic reasons – is that ‘four-sided islands’ are not allowed. To elaborate: ‘islands’ are empty bounded regions without a dot. A four-sided island appears like a diamond without a dot \diamond . Four-sided islands arise mainly when splices are made at adjacent points. So generally splices are made at alternate points [3].

Is it possible that somehow avoidance of adjacent splicing points favours choice of parallel splicing points that always split a loop? A detailed examination of a large number of splicing patterns indeed supports the conjecture. The strategy of choosing alternate points along the edges of rectangles for splicing is a convenient one, especially for large *kolams*. For small *kolams* like the 5^2 all the splicing points are adjacent as in Figure 3 (b).

To highlight the importance of the choice of splicing points we show a 15^2 coded by Q (13 1 14 15). It consists of a central square 13^2 and four linear strings 1×14 as the surrounding ‘rectangles’. The linear strings are spliced to the central square at odd numbered positions 1 3 513 indicated by dots in Figure 5(a). These dots appear between the top two rows of 15 dots in the

kolam. The result is three loops (Red, Blue and Green). When the splices are made at even numbered positions 2 4 6 14 the outcome is a single loop as in Figure 5(b). Note that there are seven sets of 4-splices in each Figure. But only one splice of a 4-splice is marked. In both the central 13^2 is Q (3 5 8 13) based on the canonical Fibonacci series.

3.0. Loop Evolution as a Binary Tree

The empirical splicing rule we adopt stipulates that an operation (splice or unsplice) in a single loop gives two loops and between two loops it yields single loop. If there are l (> 1) loops an operation results in $l - 1$ or $l + 1$ loops, a binary option. A set of four splices can be modelled as a binary tree with four nodes. This is shown in Figure 6 as ‘top-down’ tree with the root at the top. In the beginning there is a single loop at the top. After the first splice there are two loops. The second splice changes the number of loops to one or three at the next level depending on whether the splice is in a single loop or between two loops. We assume this is random and assign equal probabilities (1/2) to the two branches. If the next splice occurs in a single loop the result is two loops with probability 1. If it occurs in one of the three loops the outcome is two loops or four loops with probabilities 2/3 and 1/3 respectively. Among l loops if a random splicing point is chosen between two loops, the loops are likely to be the same loop with probability $1/l$ and different with probability $(l-1)/l$. Hence the branches $3 \rightarrow 2$ and $3 \rightarrow 4$ are assigned probabilities 2/3 and 1/3. After the last splice we end up with $l = 1, 3$ or 5 . Their relative probabilities are calculated by multiplying the probabilities assigned to each evolutionary path. For example $l = 1$ is the end-point of two pathways 1-2-1-2-1 or 1-2-3-2-1. Their respective probabilities are $(1)(1/2)(1)(1/2) = 1/4$ and $(1)(1/2)(2/3)(1/2) = 1/6$. Together they add up to $5/12$ ($=0.417$).

A similar calculation for $l = 3$ as the end-point shows three pathways 1-2-1-2-3, 1-2-3-2-3 and 1-2-3-4-3 with probabilities 1/4, 1/6 and 1/8 adding up

to $13/24$ ($= 0.542$). Finally the probability of $l = 5$ is $(1)(1/2)(1/3)(1/4) = 1/24$ ($=0.041$).

Summarising, starting from a single loop, after a set of four splices, the number of loops is 1, 3 or 5 with probabilities 0.417, 0.542 and 0.041 respectively.

It is easy to see why the final number of loops is always odd (1, 3, 5). Since the starting number of loops (one in this case) is odd every 4-splice will end in an odd number of loops. The number of splices 4 (an even number) changes the number of loops by 0, 2 or 4, i.e. 1 loop \rightarrow 1, 3, 5 loops. In general, starting from i loops the end result after any number of 4-splices is j loops with the same parity as i . In other words (i, j) are either both even or both odd. In our analysis i being odd (1 3 5) j is also odd (1 3 5).

For tracing the evolution of 4-splices we have to create three binary trees starting at the root with $i = 1, 3, 5$. The tree for $i = 1$ is already done in Figure 6. The binary tree for $i = 3$ is shown in Figure 7 (top). As for $i = 1$, here too the branches are labelled with probabilities. At the bottom of the tree, the contributors to final configuration of $j = 1, 3, 5, 7$ are shown. Starting with $i = 3$, the probability of final $j = 7$ is $(1/3)(1/4)(1/5)(1/6) = 1/360$ ($= 0.0028$) less than 0.3%. The binary tree with $i = 5$ at the apex is more complex (Figure 7, bottom). One notes a finite though small probability for $j = 9$ (0.000595). For $j = 7$, the probability is 0.0211. We wish to ignore contributions from $j = 7, 9$ but cannot drop them, since the sum of probabilities should remain 1. We therefore merge the contributions from $j = 7$ with that of $j = 5$ (probability 0.041) in Figure 6.

Figures 6 and 7 trace the evolution of a 4-splice (say like PQRS), starting with i loops ($i = 1, 3$ or 5) ending up with j loops ($j = 1, 3$ or 5). How do the loops evolve further when another 4-splice (say like KLMN) is added? Ultimately we have to deal with a large number of 4-splices. This can be done elegantly using matrices for loop probabilities.

4.0. Loop Probability Matrix

Let $p(i j)$ be the probability of initial i loops leading to final j loops after a set of 4-splices. The i, j values can be 1, 3 or 5. These probabilities define a 3×3 matrix S of nine elements

$$\begin{array}{ccc} p(11) & p(13) & p(15) \\ p(31) & p(33) & p(35) \\ p(51) & p(53) & p(55) \end{array} \quad (4)$$

Each element $p(i j)$, $i, j = 1, 3, 5$ can be calculated from the binary tree diagrams (Figures 6, 7), e.g. we have already determined $p(11) = 0.417$, $p(13) = 0.542$. The full matrix S is

$i \setminus j$	1	3	5*
1	0.417	0.542	0.041
3	0.361	0.557	0.082
5	0.200	0.570	0.230

(* these probabilities include small contributions from $j = 7, 9$)

How do the loop probabilities change after two sets of 4-splices? Let the new probabilities be $q(i j)$. For example

$$q(11) = p(11) p(11) + p(13) p(31) + p(15) p(51)$$

In general

$$q(i j) = p(i 1) p(1 j) + p(i 3) p(3 j) + p(i 5) p(5 j) \quad (5)$$

for $i, j = 1, 3, 5$.

Equation (5) is exactly the same as the rule for multiplication of matrix S by itself. Matrix $|q(i j)|$ is simply $S*S$. (* denotes matrix multiplication) [5].

In Figure 8, matrix S and matrix $S^2 (= S*S)$ are shown, the latter being the square of the first. To find the probabilities after three sets of 4-splices, multiply S^2 by S to get S^3 . After four sets of splices the loop probability matrix is S^3*S or S^4 . The matrix elements quickly converge to nearly constant values:

$$P(11) = 0.37 \quad P(13) = 0.55 \quad P(15) = 0.08$$

$P(i, j)$ is the limiting value of probability after a large number of 4-splices. Notice that the probabilities are also nearly independent of the value of i ; in other words all the rows of matrix S4 are nearly the same.

The most probable number of loops at end-point is three with a probability of 0.55 and the probability of the desired final single loop is 0.37. This leaves a small contribution of 0.08 for the probability of 5 or more loops. The relative probabilities for $j = 1$ and 3 are roughly 2:3. These results are consistent with a qualitative empirical observation that three loops are the most likely end result.

One can understand this remarkable phenomenon by referring back to the evolution of 5^2 *kolam* after a set of two 4-splices (Figure 3). The pathway for the splice set PQRS is 1-2-1-2-3 but adding the set KLMN reverses the pathway with 3-2-1-2-1. So, as the number of splice sets increases the evolution of loops is a series of alternating waxing and waning probabilities. It is surprising however, that the limiting value is very quickly attained, after a mere two or three sets of splices, as seen in Figure 8.

4.1. Loop Probability Matrix for Rectangular *Kolams*.

In the previous section only square Fibonacci *Kolams* were considered. To ensure four-fold rotational symmetry, splices were made in sets of four or 4-splices. The binary trees of Figures 6, 7 were for 4-splices.

To create rectangular Fibonacci *Kolams* with two-fold rotational symmetry, two Fibonacci Quartets are used (Section 1). Two-fold symmetry is ensured by symmetrically placed pairs of splices or 2-splices. The binary trees of loop evolution starting with $i = 1, 3, 5$ with one 2-splice are shown in Figure 9. For example, in the binary tree with $i = 3$ the first splice gives either two loops (left) or four loops (right) with relative probabilities $2/3$ and $1/3$. For the next splice, if it occurs in the left branch the outcome is $j = 1$ or 3 with equal probability ($1/2$ each); if it occurs in the right branch the result is $j = 3$ or 5 with probabilities $3/4$ and $1/4$. The final contribution to $j = 3$ comes from two

pathways 3-2-3 and 3-4-3 with probabilities 1/3 and 1/4 adding up to 7/12 (= 0.584). So $p(33) = 0.584$. The full matrix R is

$i \setminus j$	<u>1</u>	<u>3</u>	<u>5*</u>
1	0.500	0.500	0
3	0.333	0.584	0.083
5	0	0.600	0.400

(* these probabilities include small contributions from $j = 7$)

How do the probabilities change when a second 2-splice is added? As in the case of square *kolams* (Section 4.0), here we simply multiply the matrix by itself or square R to get R2. Figure 10 shows eight matrices R, R2, R3.....R8. Rn is the matrix for ‘n’ 2-splices. Each is obtained by multiplying the previous one by R: $R_n = R[n-1] * R$.

As in the square *kolams*, here too the probabilities converge to limiting values. As expected these limiting values are precisely the same as before. This is because two consecutive 2-splices make a 4-splice; the eight 2-splices are equivalent to four 4-splices (Figure 8) or $R_8 = S_4$.

4.2. Window-frame *Kolam*.

In a standard Fibonacci Rectangular *Kolam*, four rectangles are spliced to one central rectangle ($i = 5$ loops). When the central rectangle is missing we have a ‘window-frame’ an interesting variant. In Figure 11 is shown a 9 x 13 *kolam* based on a pair of quartets

$$Q_1 (5 \ 2 \ 7 \ 9)$$

$$Q_2 (7 \ 3 \ 10 \ 13)$$

[Section 1, Eqs. (2), (3)]. It is reproduced from Part III [3]. The central rectangle 5 x 7 is missing. The pairs of rectangles 2 x 10 and 7 x 3 are spliced at four 2-splices indicated by O. The result is $j = 2$ loops. Whatever be the choice of quartets or splices, the window-frame cannot be single-loop ! This is because of parity conservation (Section 3). The starting number $i = 4$ loops

(since the rectangle is omitted), i is even and each 2-splice can change i by 0 or 2 making j even. The least j value is 2.

In a nutshell, both in square and rectangular *kolams*, splicing at a large number of points ultimately results in a small odd number of loops (1, 3, 5) with relative probabilities 37 %, 55% and 8 % respectively. If the desired single loop is not achieved one or two sets of points can be unspliced to attain a single loop.

5.0. Discussion and Summary

One can estimate the maximum number of splicing points available in constructing a Fibonacci *Kolam*. Splices occur along edges of the rectangle as indicated in Figures 1, 2. Since splices are chosen at alternate points along an edge, the maximum number of splices along an edge is $\approx c/2$ for square *kolams*. Since four-fold symmetry forces splicing at four symmetrically placed points, the maximum number of splices $\approx 2c$. For Rectangular *Kolams* the maximum number of splices $\approx c_1 + c_2$ [3]. In large square *kolams* of size d^2 ($d = 20 - 30$), $c \approx 0.6 d$ and the maximum number of splices is $\approx 1.2 d$ or 24 – 36. The number of 4-splices is 6 to 9.

To track the evolution of loops through many 4-splices or 2-splices in a *kolam* an elegant matrix operation is used (Section 4.0). The basic matrix S for a 4-splice is a 3×3 matrix of probabilities $p(i, j)$. $p(i, j)$ is the probability that an initial number of i loops result in a final number of j loops after one 4-splice. To obtain the probabilities after ' n ' 4-splices 'multiply' S by itself ' $n-1$ ' times: $S_n = S * S * S * \dots * S$. The 'multiplication' is according to the rules of matrix multiplication.

It is a remarkable fact that even with ≈ 30 splices the final number of loops j is most likely 1, 3 or 5. The relative probabilities are 37 %, 55 % and 8 % (Section 4). This is because the pathways of loop evolution tend to alternately rise and fall keeping the final j to low values. The above

phenomenon is illustrated in the evolution of 5^2 *kolam* (Figure 3) and the binary tree diagrams (Figures 6, 7, 9). ‘What goes up must come down’!

The binary trees have a simple structure because of the empirical splicing rule: ‘a splice or unsplice in $l (>1)$ loops results in $l - 1$ or $l + 1$ loops.’ (Figure 4). This lends itself to a simple modelling of loop evolution. It is suggested that this simplification is a consequence of the choice of alternating splicing points along the edges to avoid four-sided islands (Section 2.2).

The number of loops in evolution conserves parity (odd/even) because splices are made in sets of 4-splices (squares) or 2-splices (rectangles). Since $i = 1$, the j values are constrained to odd values (1, 3, 5). Although we start with $i = 5$ (4 rectangles and a square), the first 4-splice merges them to a single loop (as for example in Figure 3c). So effectively the construction starts from $i = 1$ (Section 3.0). In the ‘window-frame’ *kolam* where the central rectangle is omitted, $i = 4$ and parity conservation determines the smallest j as 2.

An overview of all the Fibonacci *Kolam* work done up to the end of 2015 is presented in [6].

When I started my work on Fibonacci *Kolams* in 2008, faced with a large number of splicing choices, I settled on alternating points. The other choice was that the *kolam* has a single loop. Both choices were guided by aesthetic considerations. Most *kolams* I have seen in traditional books are multiple loops because they are easier to draw. Single loop large *kolams* are harder to achieve. Indeed ‘*anthadhi kolams*’ (*kolams* without end or beginning) are considered special.

Traditional *kolams* have no restrictions on island shapes; four-sided islands occur in plenty. Intuitively I felt they may be excluded. A *kolam* when covered by a grid of unit squares will be made up of ‘unit cells’ with a dot in the centre and a shape made up of line/s embedded in it. The four-sided islands masquerade as empty unit cells without a central dot.

In preparation:

Basic principles underlying Fibonacci *Kolams* are well understood and the rules of splicing/unsplicing the parts into a whole (single loop) are simple. However for large *kolams* experimenting with multiple choices of splicing can be very time-consuming. Clearly computer-aided *kolam* drawing is desired.

Analysing a large number of *kolams* for their basic structure of square unit cells each with a central dot, one finds that almost all *kolams* have unit cells that are of eight basic shapes. However rotations (4) and reflections (4) of the basic shapes lead to 31 distinct variations. The eight basic shapes have been drawn on the computer and each of 31 variants is assigned a ‘macro’. This makes it easy to draw large *kolams* by typing the macros. This is a major step forward in computer-aided drawing of *kolams* [7].

P.S. I conclude with some personal thoughts. When I figured out matrix multiplication as a natural tool for the study of evolution of loops in January 2015, it was an exciting moment. Suddenly, juggling with a large number of splicing points in a large *kolam* seemed tractable. The ‘juggling’ reminded me of a lecture at the Allahabad session of the Indian Science Congress in 1949 in which the speaker compared matrix operations with juggling of balls, simultaneously handling a large number of objects as they evolve in time. As a first year M.Sc. (Physics) student at BHU, I was merely aware of matrices but the evocative analogy aroused my deeper interest in matrices. These thoughts occurred on 18 June 2016 when I completed this paper. Next day, I read in The Hindu that the day was celebrated all over the world as World Juggling Day [8]!

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Figure Captions

- Figure 1. Construction of Square Fibonacci *Kolams*
- Figure 2. Construction of Rectangular Fibonacci *Kolams*
- Figure 3. Evolution of Loops (5^2)
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(b) splices at even-numbered positions.
- Figure 6. Binary tree of evolution from a single loop (square *kolams*)
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(c) 5 loops (rectangular *kolams*)
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- References
- Figures
- Figure captions.

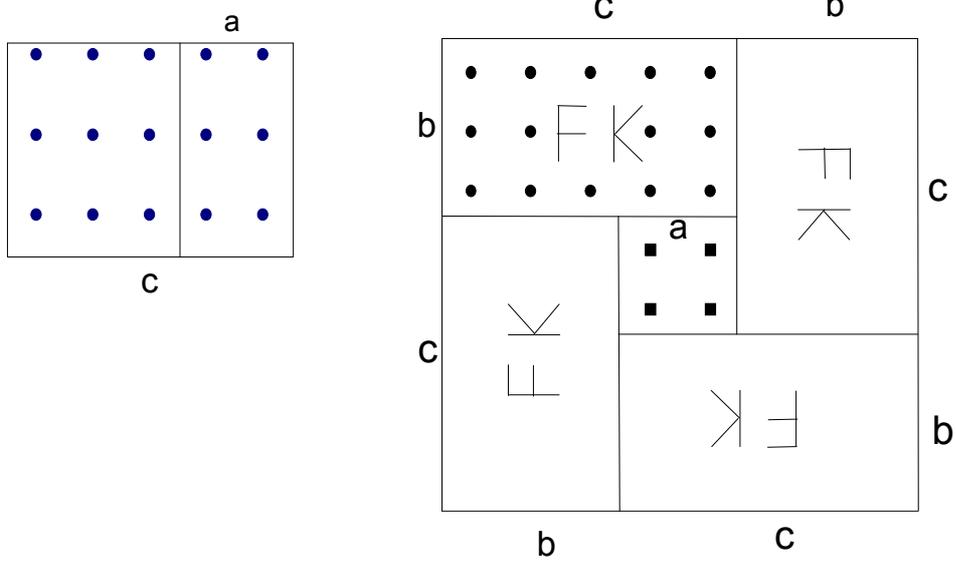
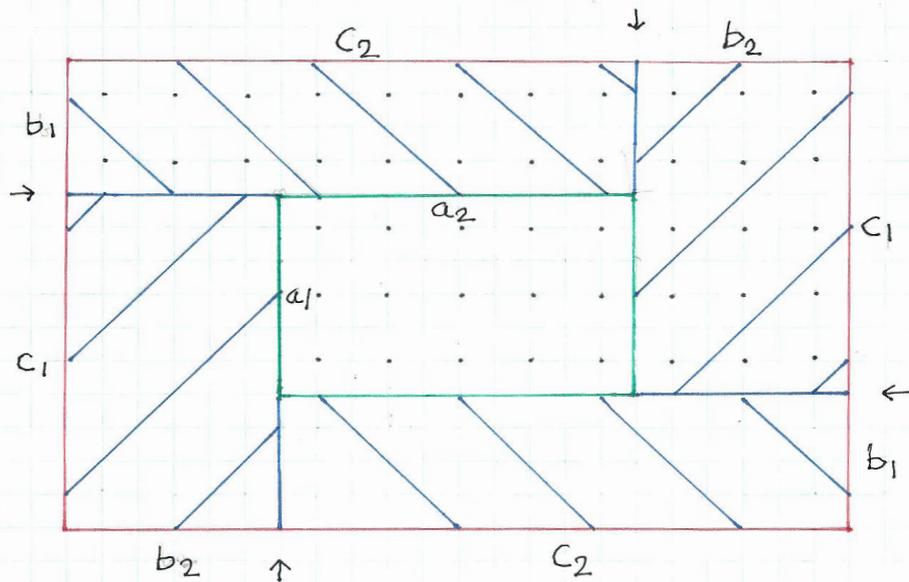


Figure 1



$$Q_1(a_1, b_1, c_1, d_1) \quad c_1 = a_1 + b_1, \quad d_1 = c_1 + b_1$$

$$Q_2(a_2, b_2, c_2, d_2) \quad c_2 = a_2 + b_2, \quad d_2 = c_2 + b_2$$

$$d_1 d_2 = a_1 a_2 + 2b_1 c_2 + 2c_1 b_2$$

Construction of Rectangular Kolams

Figure 2

Evolution of Loops

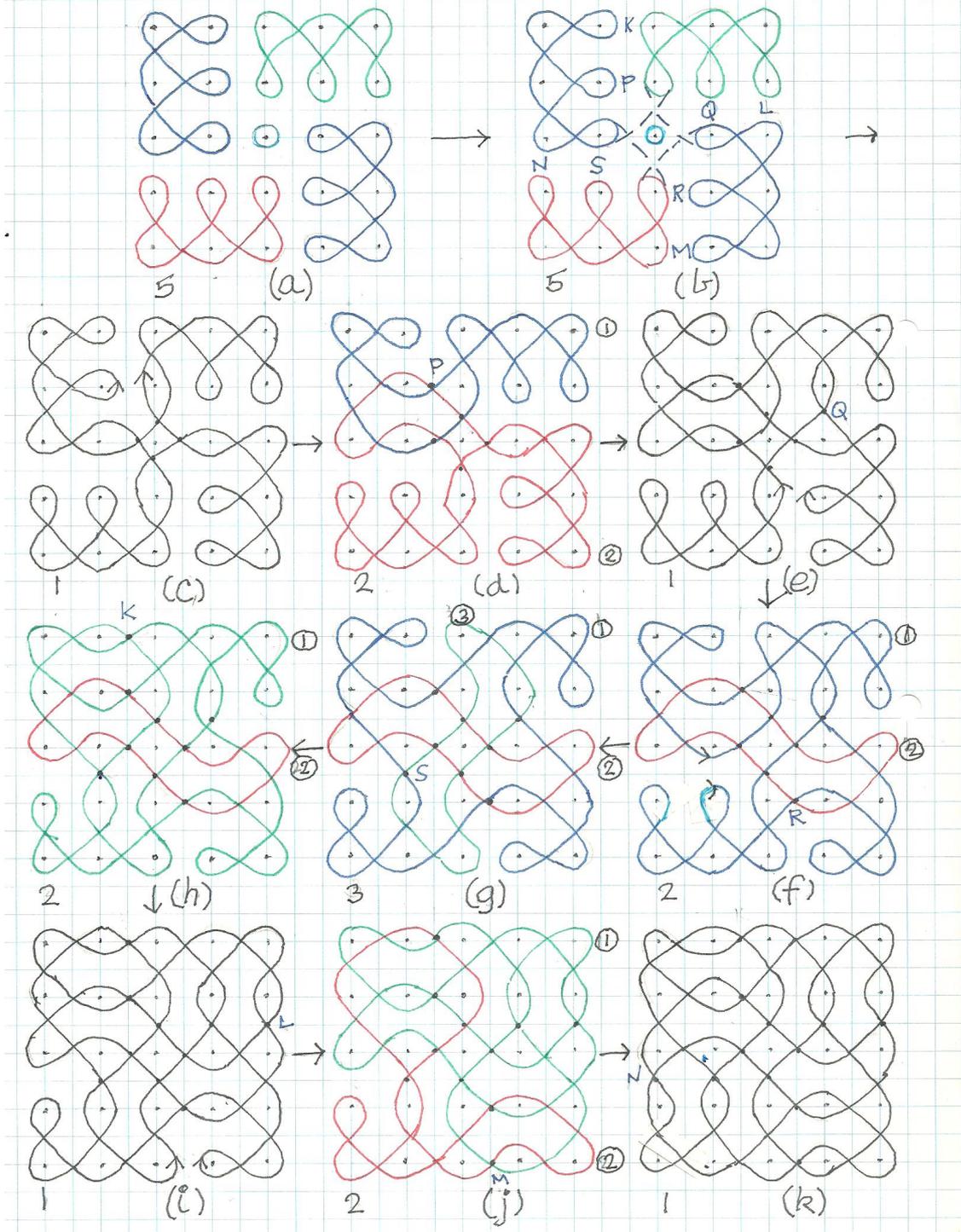


Figure 3

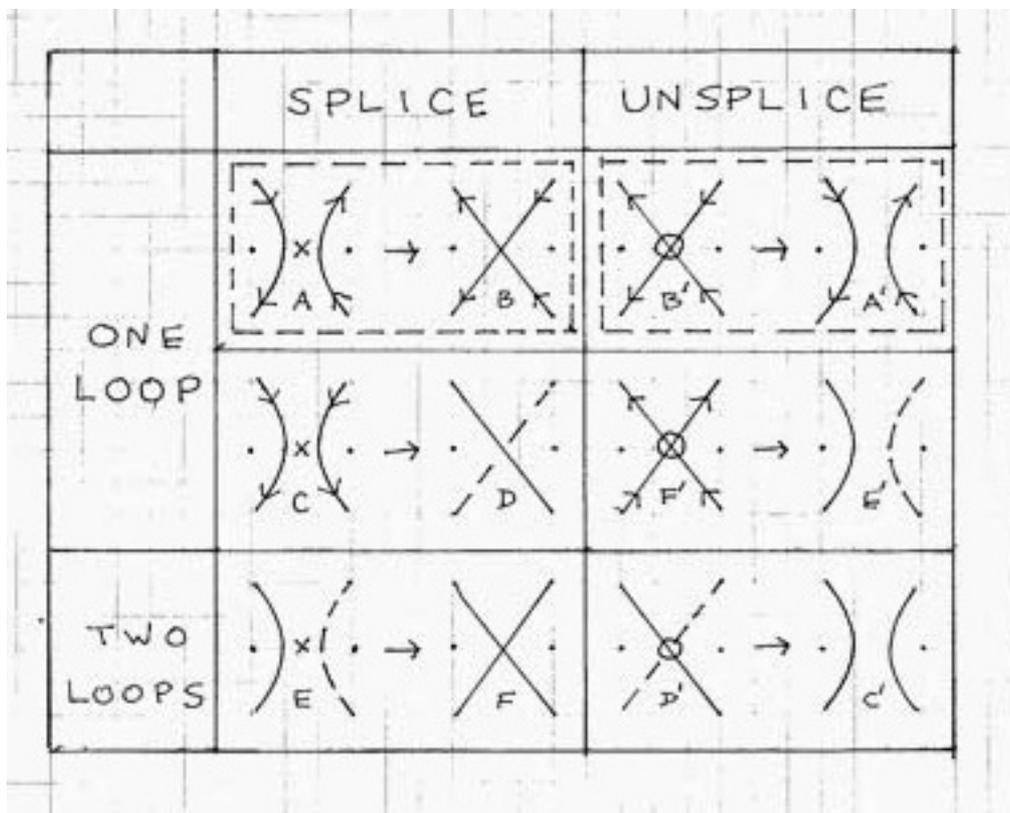


Figure 4

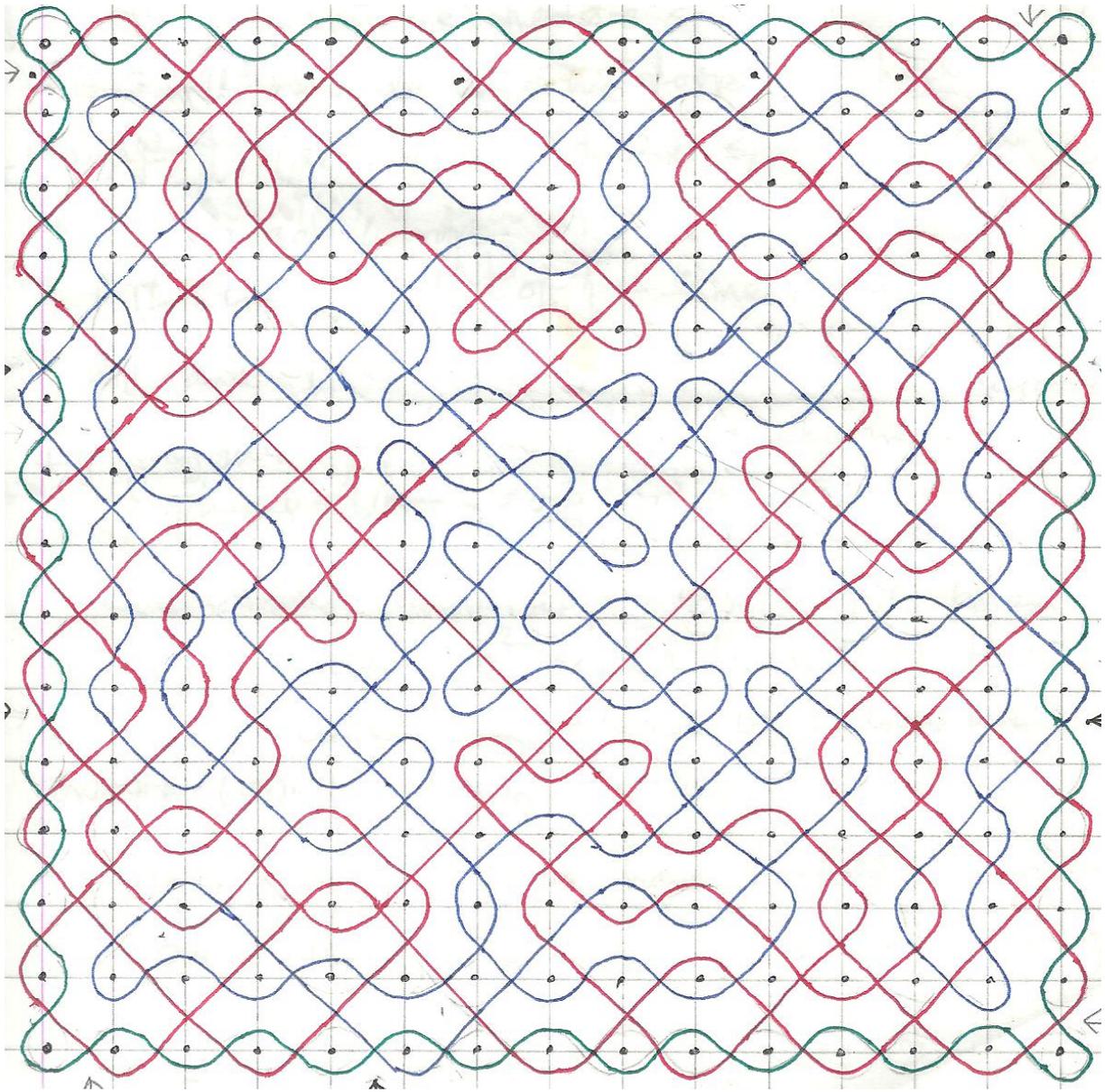


Figure 5(a)

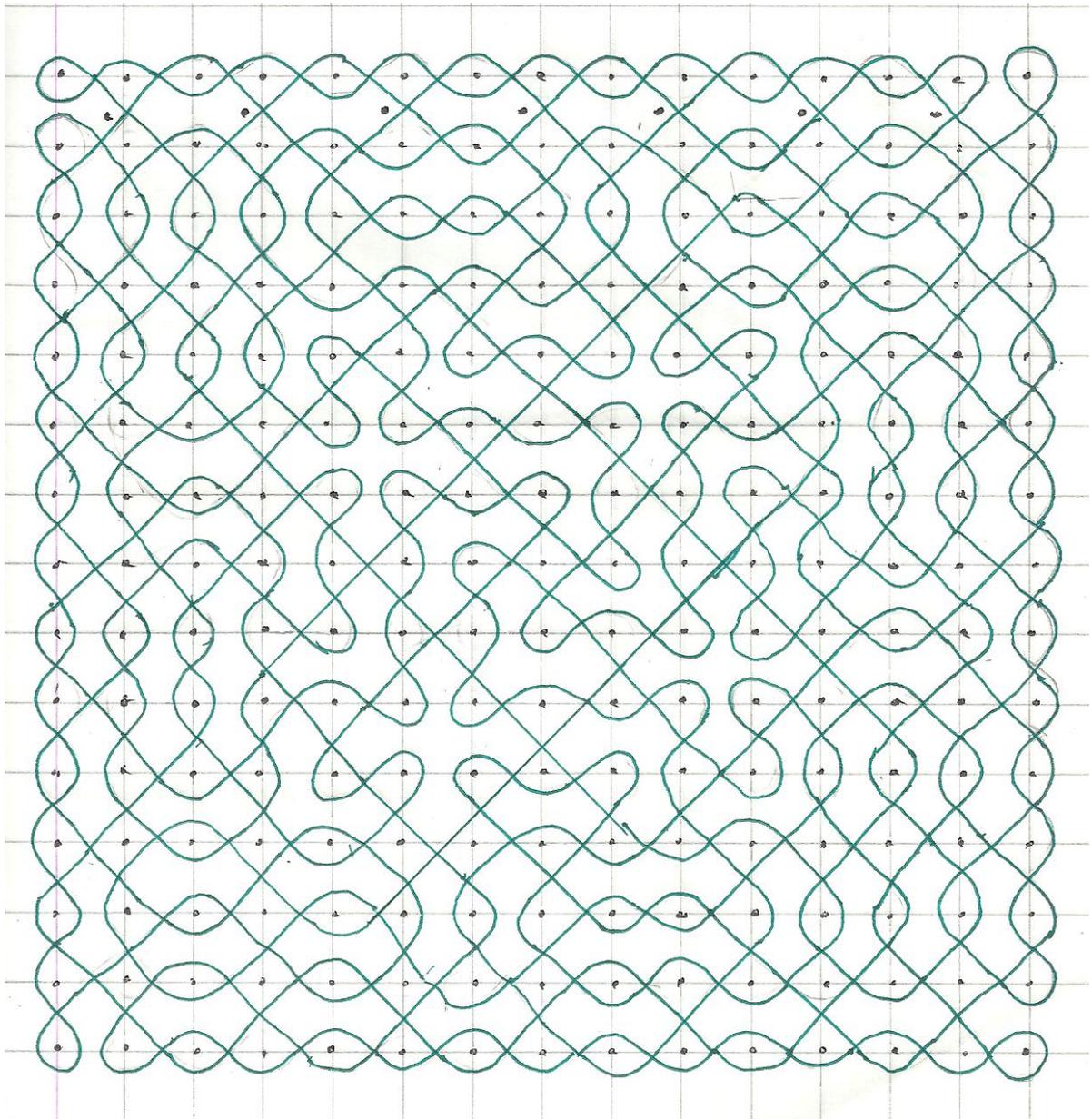


Figure 5 (b)

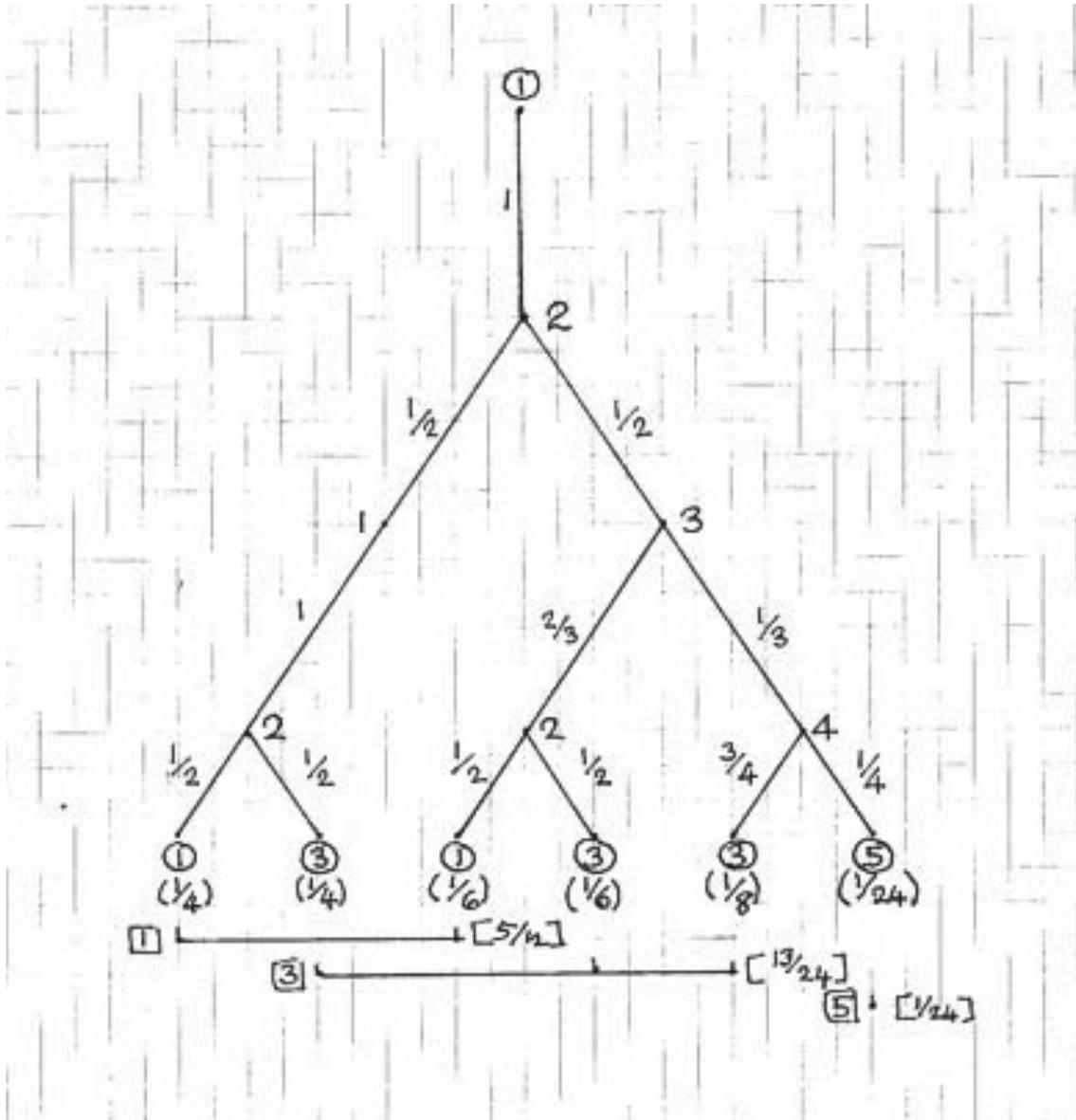


Figure 6

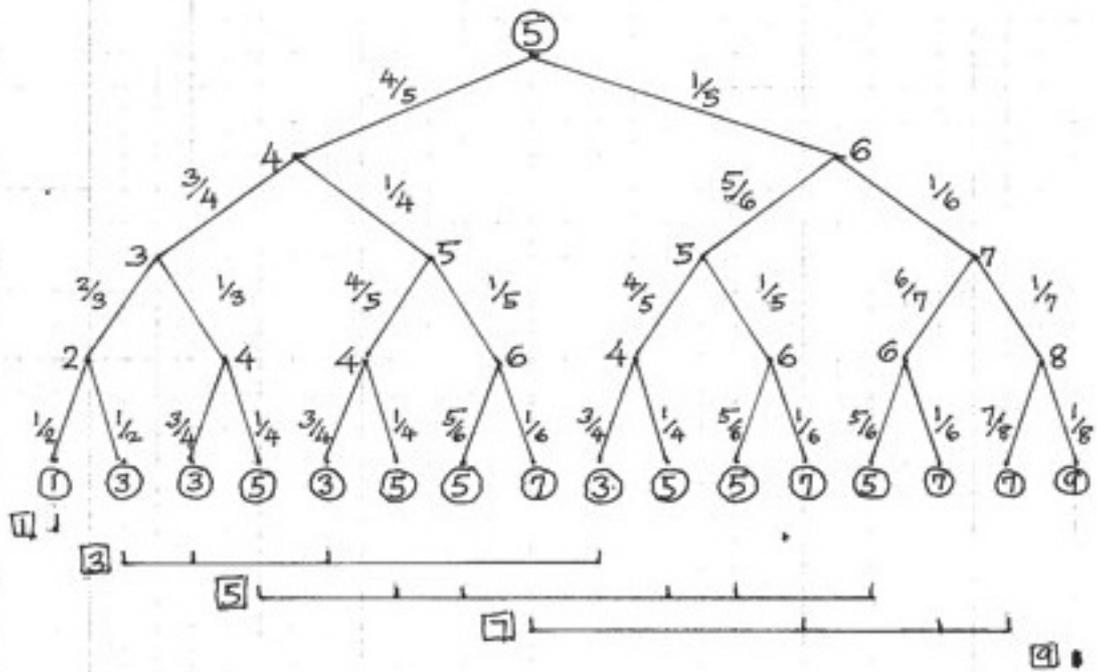
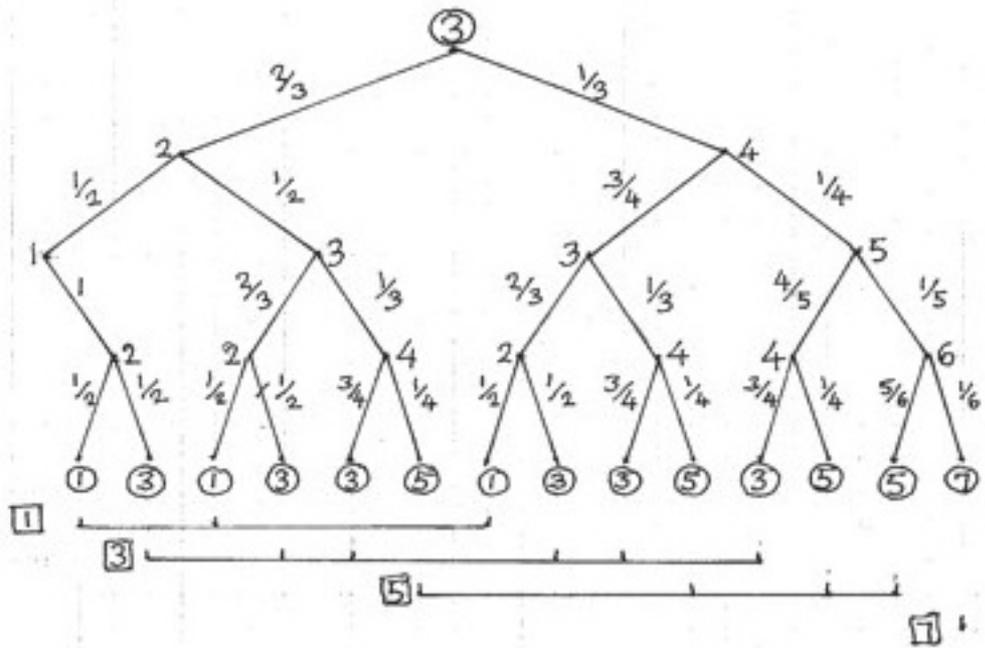


Figure 7

**EVOLUTION OF LOOPS:
SQUARE FIBONACCI KOLAMS WITH 4-FOLD SYMMETRY**

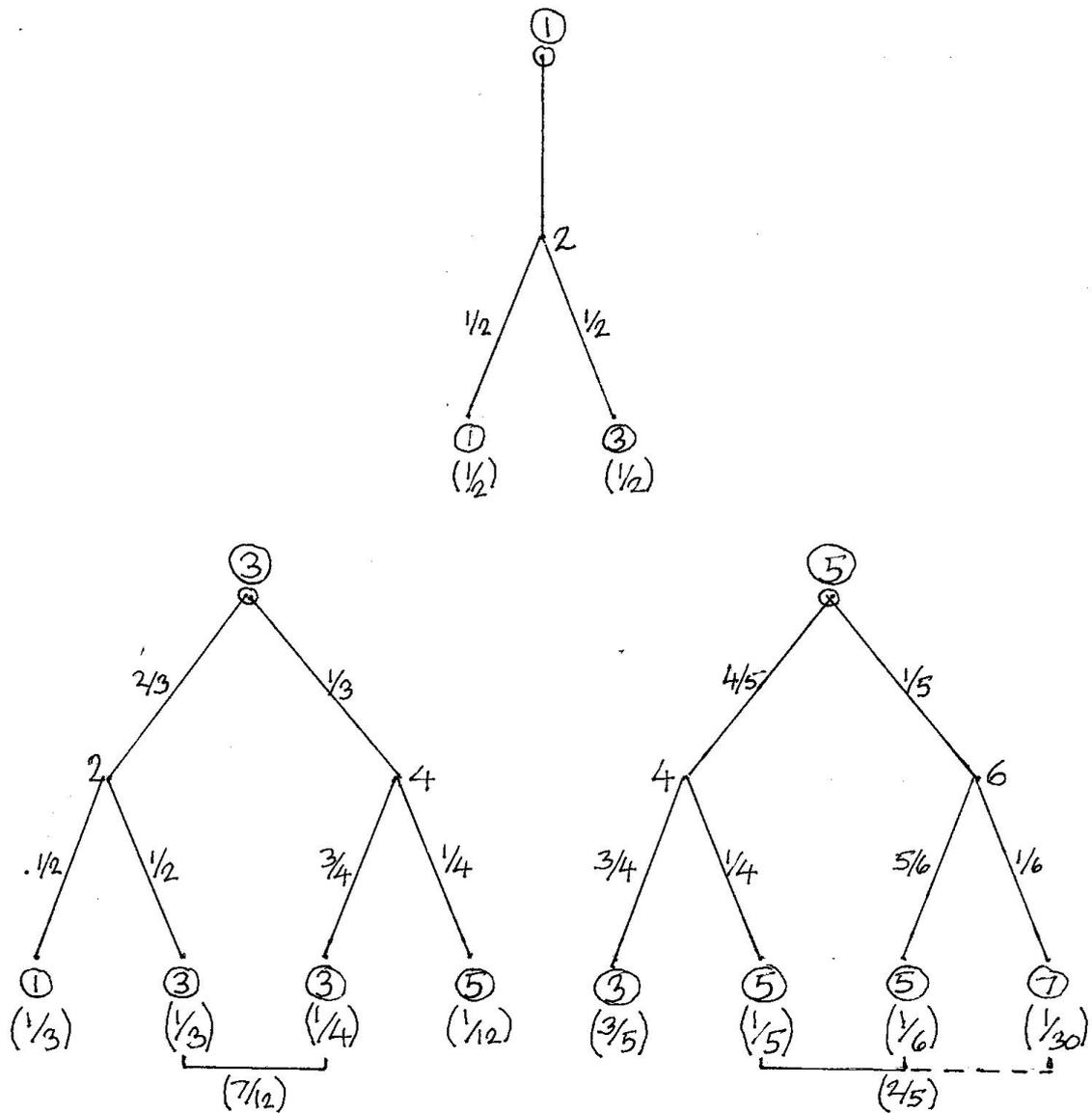
<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px 5px;">I\J</th> <th style="padding: 2px 5px;">1</th> <th style="padding: 2px 5px;">3</th> <th style="padding: 2px 5px;">5</th> <th style="padding: 2px 5px;"></th> </tr> </thead> <tbody> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px; text-align: center;">0.417</td> <td style="padding: 2px 5px; text-align: center;">0.542</td> <td style="padding: 2px 5px; text-align: center;">0.041</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;">S</td> <td style="padding: 2px 5px;">3</td> <td style="padding: 2px 5px; text-align: center;">0.361</td> <td style="padding: 2px 5px; text-align: center;">0.557</td> <td style="padding: 2px 5px; text-align: center;">0.082</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;"></td> <td style="padding: 2px 5px;">5</td> <td style="padding: 2px 5px; text-align: center;">0.200</td> <td style="padding: 2px 5px; text-align: center;">0.570</td> <td style="padding: 2px 5px; text-align: center;">0.230</td> <td style="padding: 2px 5px;">1</td> </tr> </tbody> </table>	I\J	1	3	5		1	0.417	0.542	0.041	1	S	3	0.361	0.557	0.082	1		5	0.200	0.570	0.230	1	<table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px 5px;">I\J</th> <th style="padding: 2px 5px;">1</th> <th style="padding: 2px 5px;">3</th> <th style="padding: 2px 5px;">5</th> <th style="padding: 2px 5px;"></th> </tr> </thead> <tbody> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px; text-align: center;">0.3778</td> <td style="padding: 2px 5px; text-align: center;">0.5513</td> <td style="padding: 2px 5px; text-align: center;">0.0710</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;">S2</td> <td style="padding: 2px 5px;">3</td> <td style="padding: 2px 5px; text-align: center;">0.3680</td> <td style="padding: 2px 5px; text-align: center;">0.5527</td> <td style="padding: 2px 5px; text-align: center;">0.0793</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;"></td> <td style="padding: 2px 5px;">5</td> <td style="padding: 2px 5px; text-align: center;">0.3352</td> <td style="padding: 2px 5px; text-align: center;">0.5570</td> <td style="padding: 2px 5px; text-align: center;">0.1078</td> <td style="padding: 2px 5px;">1</td> </tr> </tbody> </table>	I\J	1	3	5		1	0.3778	0.5513	0.0710	1	S2	3	0.3680	0.5527	0.0793	1		5	0.3352	0.5570	0.1078	1
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EVOLUTION OF LOOPS: S(I,J) IS THE BASIC MATRIX.
 ELEMENT (I,J) IS THE PROBABILITY THAT
 STARTING WITH I LOOPS, AFTER A SET OF FOUR SPLICES
 THE RESULT IS J LOOPS. I,J = 1, 3, 5.
 S2(I,J), S3(I,J), S4(I,J) ARE PROBABILITY MATRICES AFTER
 RESPECTIVELY 2,3,4 SETS OF SPLICES.
 THE MATRICES QUICKLY CONVERGE TO
 A FIXED MATRIX WITH IDENTICAL ROWS. SEE TEXT

NOTE THE FOLLOWING EQUALITIES

$$\begin{aligned}
 R2 &= S \\
 R4 &= S2 \\
 R6 &= S3 \\
 R8 &= S4
 \end{aligned}$$

Figure 8



Binary Tree of Evolution of Loops (Rectangular Kolams)

Figure 9

**EVOLUTION OF LOOPS:
RECTANGULAR FIBONACCI KOLAMS WITH 2-FOLD SYMMETRY**

R	I\J	1	3	5	1	R5	I\J	1	3	5	1
	1	0.500	0.500	0	1		1	0.3726	0.5534	0.0739	1
	3	0.333	0.584	0.083	1		3	0.3686	0.5543	0.0771	1
	5	0	0.600	0.400	1		5	0.3560	0.5572	0.0868	1
R2	I\J	1	3	5	1	R6	I\J	1	3	5	1
	1	0.4165	0.5420	0.0415	1		1	0.3706	0.5539	0.0755	1
	3	0.3610	0.5574	0.0817	1		3	0.3689	0.5543	0.0768	1
	5	0.1998	0.5904	0.2098	1		5	0.3636	0.5555	0.0810	1
R3	I\J	1	3	5	1	R7	I\J	1	3	5	1
	1	0.3887	0.5497	0.0616	1		1	0.3697	0.5541	0.0762	1
	3	0.3661	0.5550	0.0789	1		3	0.3690	0.5542	0.0767	1
	5	0.2965	0.5706	0.1329	1		5	0.3668	0.5548	0.0785	1
R4	I\J	1	3	5	1	R8	I\J	1	3	5	1
	1	0.3774	0.5523	0.0703	1		1	0.3694	0.5542	0.0765	1
	3	0.3679	0.5545	0.0776	1		3	0.3691	0.5542	0.0767	1
	5	0.3383	0.5612	0.1005	1		5	0.3681	0.5544	0.0774	1

EVOLUTION OF LOOPS: $R(I,J)$ IS THE BASIC MATRIX. ELEMENT (I,J) IS THE PROBABILITY THAT STARTING WITH I LOOPS, AFTER A PAIR OF TWO SPLICES THE RESULT IS J LOOPS. $I,J = 1, 3, 5$.
 $R2(I,J), R3(I,J), R4(I,J), \dots$ ARE PROBABILITY MATRICES AFTER RESPECTIVELY 2,3,4 ...PAIRS OF SPLICES. THE MATRICES QUICKLY CONVERGE TO A FIXED MATRIX WITH IDENTICAL ROWS. SEE TEXT

Figure 10

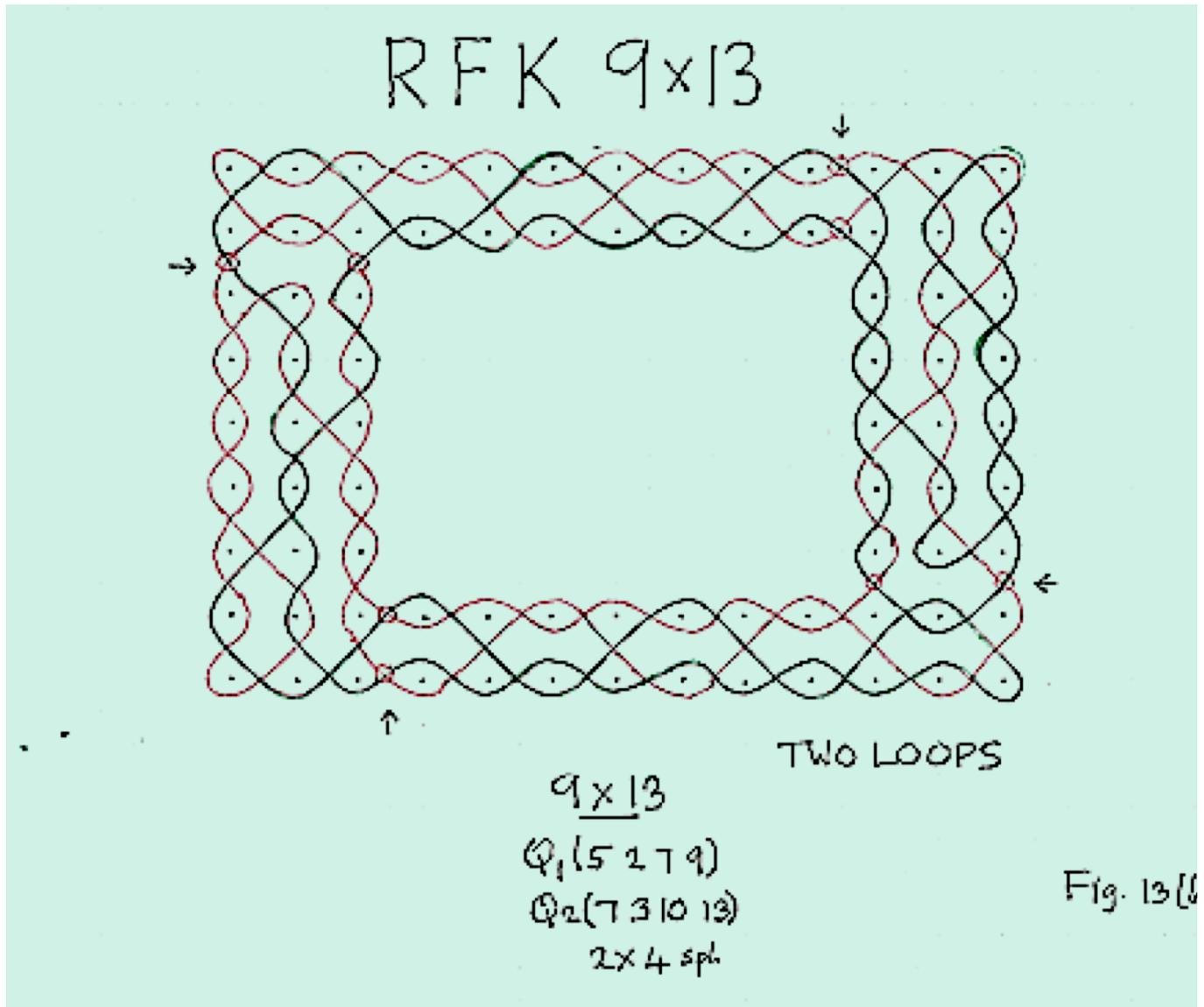


Figure 11