A VARIANT OF KAPREKAR ITERATION AND KAPREKAR CONSTANT

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INTRODUCTION.

First we review the Kaprekar Iteration leading to the Kaprekar Constant 6174. Then we present a variation of Kaprekar Iteration that leads to some interesting new results.

Kaprekar Iteration.

Step 1. Pick a 4-digit number (not all digits same).

Step 2. In the above rearrange the digits to get the largest 4-digit number (L) and the smallest 4-digit number (S). Find X = L – S to obtain another 4-digit number. (Add leading 0's if needed).

Step 3. Repeat step (2) until X repeats.

Consider an example.

Step 1. 2008

Step 2. L = 8200, S = 0028

X1 = 8200 – 0028 = 8172.

Step 3. X2 = 8721 – 1278 = 7443

X3 = 7443 – 3447 = 3996

X4 = 9963 – 3699 = 6264

X5 = 6642 – 2466 = 4176

X6 = 7641 – 1467 = **6174**

X7 = 7641 – 1467 = 6174 = X6

The successive Kaprekar Iterates are X1 to X6. Iterates beyond X6 are the same as X6. X6, the number 6174 is a *fixed point*. It is reached after 6 iterations of the seed number 2008.

Step 2 is called the Kaprekar Iteration (KI) and 6174 is known as the Kaprekar Constant (KC). It is clear why "rep-digit" numbers or rep-numbers (e.g. 1111, 2222 ...9999) are excluded in Step 1. For such numbers L = S and X = L – S = 0.

Some remarkable properties of KI and KC are the following. (1) 6174 is the only fixed point. In other words, all 4-digit numbers subjected to repeated KI lead to 6174. (2) It requires no more than 7 iterations to reach 6174 whatever be the starting number (the seed). Considering that there are 8991 possible 4-digit numbers (excluding 9 rep-numbers), it is remarkable that 6174 is the only fixed point and the maximum number of iterations is 7.

Modified Kaprekar Iteration.

The proposed variation in KI is the following. In Step 2 of KI, X = L-S. In Modified Kaprekar Iteration (MKI), we find L+S and L-S. *If L+S is a 4-digit number with not all its digits the same, then X = L+S. Otherwise X = L-S.* How does repeated MKI modify the results of repeated KI ? Let us consider the same example of the seed number 2008.

Step 1. 2008

Step 2. L = 8200, S = 0028

X1 = 8200 + 0028 = 8228

Step 3. X2 = 8822 – 2288 = 6534

X3 = 6543 – 3456 = 3087

X4 = 8730 + 0378 = 9108

X5 = 9810 – 0189 = 9621

X6 = 9621 – 1269 = 8352

X7 = 8532 – 2358 = 6174

X8 = 7641 + 1467 = 9108 = X4.

MKI terminates in a *cycle of numbers*

$$9108 \rightarrow 9621 \rightarrow 8352 \rightarrow 6174 \qquad \text{(A)}$$

that repeats. This "limit cycle" of period 4, replaces the fixed point 6174 of KI. In MKI it takes 4 iterations (X1 to X4) starting from 2008 to reach the period 4 limit cycle. In contrast in KI, it takes 6 iterations starting from 2008 to reach a fixed point (6174). The fixed point can be considered as a cycle of period one. We can ask the following questions.

(1) Is 6174 excluded as a fixed point in MKI for all seeds? Yes. Suppose one of the iterates is 6174; then the next iterate will be L+S (7641 +1467 = 9108) a 4-digit number and not L-S = 6174. So for all seeds, 6174 of KI is replaced by the limit cycle (A).

(2) Is there some other fixed point or some other cycle? No. This assertion is based on tracing the paths of all seeds to their destination. All reach the cycle (A). The maximum number of iterations is 6.

Summarizing, for 4-digit numbers repeated MKI always leads to the limit cycle (A) (9108→9621→8352→6174) after no more than 6 iterations.

MKI for 2-digit numbers.

We first review the KI for 2-digit numbers. Repeated KI starting from a 2-digit number (the seed) terminates in a cycle of 5 numbers

$$09 \rightarrow 81 \rightarrow 63 \rightarrow 27 \rightarrow 45 \qquad \text{(B)}$$

There are 81 possible seed numbers (excluding 11, 22, 33 . . .99), and each of them enters the cycle (B) at one of 5 numbers. It requires no more than one iteration to reach the cycle. As an example, for the seed number 46, X1 = 64-46 = 18, which expressed in "L form" is 81 in cycle (B).

MKI repeated for 2-digit numbers also leads to the cycle (B). It is interesting that no 'plus' iteration is allowed because it always leads to a rep number 11,22,33 ....99. So only minus iterations are valid as in KI. This can be proved easily as follows. Let the seed number be '*ab*' ($a > b$). The numerical value of L is *'10a+b'* and of S is *'10b+a'*. The numerical value of L+S is *'11(a+b)'* a multiple of 11 (11,22,33 ...99), all rep numbers.

For two digit numbers "MKI" = "KI".

MKI for 3-digit numbers.

As before we recall the KI for 3-digit numbers. Repeating KI for a 3-digit seed number terminates in a fixed point 495. (Check: 954-459 = 495). There are 891 seed numbers, all of which lead to the same fixed point 495 in no more than 6 iterations.

MKI too leads to the same fixed point 495. It is easy to see this. When 495 is reached at any step in MKI, the next iteration cannot be a 'plus' iteration because 954 + 459 = 1413, which exceeds 3 digits. So the next iteration has to be a 'minus' iteration leading to the fixed point 495. So 495 remains as a fixed point in MKI too. Now we ask are there any other fixed points or cycles? The answer is no. A 'brute force' approach, tracing the paths from all the 891 seed numbers, shows that the end point is always 495. It turns out that under MKI, the 3-digit case is harder to analyze than the 4-digit case.

In MKI, the fixed point is reached in 10 or less iterations. In KI, it requires 6 or less iterations. As an example of a seed requiring 10 iterations:

100→*101*→*121*→*323*→*565*→099→891→792→693→594→495 (C)

The first 4 are 'plus' iterations (shown in italics) and the remaining 6 are 'minus' iterations. Under KI, it takes only 6 iterations to reach 495. Since both KI and MKI lead to the same fixed point, although with different number of iterations, *KI and MKI are equivalent.*

MKI for numbers with greater than 4 digits.

In tracing the evolution of iterates in MKI, we have taken a known fixed point or cycle in KI and examined its modification under MKI. For 3-

digit numbers, the KI fixed point 495 remains unaffected under MKI (page 4) whereas for 4-digit numbers the KI fixed point 6174 is replaced by a cycle (A) under MKI (page 3).

Similarly we have looked at the KI fixed points and cycles with number of digits $n > 4$, up to 10. For $n = 5, 7$ there are no fixed points, but there are 3 cycles in $n = 5$ and one cycle in $n = 7$. These cycles remain unchanged under MKI. For $n = 9$, there are 2 KI fixed points and a cycle; all are unaffected by MKI. From these facts we conclude that for $n = 3, 5, 7, 9$, KI fixed points and cycles remain in tact under MKI. In all the four cases n is odd. There is good evidence to support the conjecture that this is true for *all* odd n. For $n = 4,6,8,10$, all even numbers, KI has one or more fixed points. Only one fixed point in each n changes under MKI to a cycle and the other fixed points remain. As for KI cycles, $n = 4$ has none, $n = 6$ has one and $n = 8$ has 2 cycles; under MKI all of them are destroyed and replaced by new cycles. For $n = 10$, which has 5 cycles, only one survives under MKI. This feature – most KI cycles being destroyed under MKI – is likely to be true for *all* even n.

We have seen how the known fixed points and cycles are affected by MKI; some of them are replaced by new cycles. Now we ask, can MKI yield new fixed points and cycles in addition to those derived above as modifications of KI? We have a definite answer for fixed points; *there can be no new fixed points*. To show this, suppose the new fixed point is Y_k implying that the next iterate $Y_{k+1} = Y_k$. This is possible only as a result of a 'minus' iteration, since a 'plus' iteration will always yield $Y_{k+1} > Y_k$. Now consider Y_k as a seed for KI; no iteration is required to reach the fixed Y_k ! This means that Y_k would have been found in KI too; it is not a new fixed point under MKI. This is 'proof by contradiction'. Such a proof does not work for cycles. Indeed, MKI can in principle generate new cycles unrelated to the KI cycles.

There is an important distinction between n =2, 3, 4 and all $n > 4$. For n = 2, 3, 4 we established by exhaustive search that there are no other cycles besides the ones found. But for $n > 4$, we are unable to do an exhaustive search, leaving open the possibility of undiscovered cycles under MKI.

As n increases, the number of fixed points and the number of cycles grow rapidly and nearly exponentially. Data are available for n up to 50 (http://kaprekar.sourceforge.net/). The growth rate is different for odd and even n. For n =50 the number of cycles ≈ 5200 and for n = 51 the number ≈ 2250! In contrast, data on the maximum number of iterations required to reach the end point is available only for small n. The maximum number of iterations is 19 for n = 8 for a cycle of period 3.

Summary and Conclusions

In Kaprekar Iteration (KI), successive iterates are obtained by taking the *difference* of the largest and smallest permutations of an n-digit number. Here we consider a variant of KI, called the Modified Kaprekar Iteration (MKI). In MKI, first the *sum* of the largest and smallest permutations is found and it is rejected only if the number of digits exceeds n or if it is an n-digit rep number. In both cases, the difference is taken, as in KI.

The effect of MKI on the fixed points and cycles observed in KI, for n = 2, 3, 4.....10 can be summarized as follows.

(1) For n = 2, MKI is the same as KI, since plus iterates are not valid. So, MKI is identical to KI: *"MKI"="KI"*.

(2) For n = 3, in both KI and MKI, the fixed point 495 is the same, but the maximum number of iterations is 6 in KI and 10 in MKI. We recognize this distinction by stating that *MKI is equivalent to KI.*

(3) For *n* = 5,7,9 MKI is equivalent to KI as for *n* = 3. It is conjectured that this is true for *all* odd *n*. There is, however a distinction to be made from *n* = 3. No exhaustive search has been made for new cycles under MKI for *n* > 4. It can be shown that MKI has no new fixed points, besides those found in KI.

(4) For *n* = 4, 6, 8, 10 one fixed point in each *n* is changed to a cycle, but other fixed points remain. However, most cycles are destroyed and replaced by new cycles. Therefore, *"MKI" ≠ "KI"*. This feature also is conjectured to be a general feature of *all* even *n*.

The idea of exploring the changes in Kaprekar Iteration (KI) by proposing the Modified Kaprekar Iteration (MKI), has yielded some interesting new facts. The most striking difference is in the cases of odd *n* and even *n*. It can be mainly attributed to some special properties of "L-S" iterates. *L-S is always a multiple of 9; further when n is odd, the central digit is always 9.* For odd *n* the next iteration will have L starting with 9 as the most significant digit, making it easy for "L+S" to exceed *n* digits. So "L+S" is not a valid option. This accounts for the relative stability of KI fixed points and cycles under MKI for odd *n*. In contrast, for even *n* the central digit in L-S can be any number. This explains why KI cycles are easily disrupted. Even if one member of the cycle admits a 'plus' iteration, it will break the cycle. No chain is stronger than its weakest link! This is how most KI cycles disappear under MKI for even *n*.

"L+S" iterates are always multiples of 11 when n is even. This enhances considerably the chance of their being rep numbers, which are invalid. This helps in preserving the KI fixed points under MKI. The above examples help to partly demystify some of the consequences of MKI.

It is known that in KI, the number of fixed points and cycles increase exponentially with *n.* At large *n,* cycles of period 3 dominate. It will be interesting to see how the growth pattern changes under MKI. It is found, for example, that almost all the cycles of period 3 are destroyed under MKI, drastically reducing the growth rate of the number of cycles as *n* increases. This is true for both odd and even *n*. Further systematic research, searching for patterns in fixed points and cycles, both in KI and MKI, is desirable. It is also necessary to explore if MKI yields any new cycles for odd $n > 3$.

For more details on Kaprekar Iteration see the companion article in this homepage: http://vindhiya.com/snaranan/kc/index.htm.

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Note: Following this article is an Appendix (3 pages), with interesting facts about the statistical distribution of the number of iterates needed to reach the end, starting from a seed. It makes references to this article. But this article stands on its own, independent of the Appendix.

APPENDIX

To the article

A VARIANT OF KAPREKAR ITERATION AND KAPREKAR CONSTANT

Some readers may be interested in details about the number of iterations (*m*) in Kaprekar Iteration (KI) and Modified Kaprekar Iteration (MKI) that are needed to reach the end starting from a seed. The *maximum* number of iterations required is given for KI with number of digits $n = 2, 3, 4$; they are 1,6,7 respectively. For MKI the corresponding values are 1, 10, 6. The minimum number of iterations is 0 in all *n* if the seeds happen to be the end point or members of a cycle. For example in KI, $n = 4$, a seed 6174 requires no iterations ($m = 0$) because it is also the end point.

But what is the statistical distribution of m **among** the seeds? Before discussing this, it is desirable to make a minor change in the choice of seeds. For $n = 4$, for example we have restricted the seeds to 4-digit numbers that are not rep numbers (Step 1, p1). In subsequent iterates we add 0's to make them 4 digits (Step 2). It is obvious that a seed number can in principle be *any number less than 4 digits* and leading 0's can be added to it to make it a 4-digit number. There are 9990 such numbers, all of them leading to 6174 in ≤ 7 iterations. (Compare with the 4-digit numbers 8991, p3). The same change can be made for any *n*; then Step 1 should be modified as:

Step 1. Pick a number with $\leq n$ digits (not all digits same).

This change does not affect the iteration process itself; it only increases the number of seeds by one-ninth (about 11%).

Table 1

DISTRIBUTION OF THE ITERATION LENGTHS IN KI AND MKI

4-DIGIT		
m	KI	MKI
	N(m)	N(m)
0	1	4
1	383	758
2	576	4498
3	2400	2474
4	1272	1656
5	1518	500
6	1656	100
7	2184	0
TOT	9990	9990
mean m	*4.668*	*2.693*
st dev s	*1.776*	*1.058*

TOT EXCLUDES THE
10 REP #S 0000 …9999

3-DIGIT		
m	KI	MKI
	N(m)	N(m)
0	1	1
1	149	74
2	144	120
3	270	189
4	222	180
5	150	153
6	54	132
7	0	114
8	0	21
9	0	3
10	0	3
TOT	990	990
mean m	*3.241*	*4.223*
st dev s	*1.421*	*1.888*

TOT EXCLUDES THE
10 REP #S 000 …999

The distribution in *m,* the iteration length (or number of iterations) is presented for $n = 3, 4$ for KI and MKI in Table 1. Results for KI are known (see for example *R.W. Ellis and J.R. Lewis, 2002, 'Investigations into the Kaprekar Process' accessed in Google through 'Kaprekar Constant 6174'*). We computed the distributions for MKI. *N(m)* is the number of seeds requiring *m* iterations to the end. Note that for $n = 4$, the end is the fixed point 6174 in KI and the cycle (A) in MKI (p3). For $n = 3$, the end is the fixed point 495 in both KI and MKI. For $n = 2$, KI and MKI are the same, both ending in the 5-member cycle (B).

For $n = 4$ the most probable value of m is 3 (2400) in KI and 2 (4498) in MKI. In MKI nearly half of all seeds require only 2 iterations. Another distinguishing feature is that in KI, $N(m)$ increases with m (except for the 'spike' at $m = 3$), reaching 2184 at $m = 7$. Whereas in MKI, the distribution has one maximum at $m = 3$ and gradually tapers off to 0 for $m = 7$. The mean and the standard deviation of the distributions are given in Table 1. Both are less in MKI compared to KI.

For $n = 3$, the differences in the shape of the distributions in KI and MKI are less pronounced; both have one maximum at $m = 3$ with a longer tail for MKI. Both the mean and standard deviation are greater in MKI compared to KI. This is in contrast to the case $n = 4$.

For $n = 2$, MKI is identical to KI; both have $N(m=0) = 5$ and $N(m=1) = 85$, total seeds 90.

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