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Kolam designs based on Fibonacci series

In this article S. Naranan, T.V. Suresh and Swarna Srinivasan demonstrate the connection between the beauty of Indian kolams with the elegance of Fibonacci numbers. Kolams are decorative designs in South Indian folk art. They are drawn as lines and curves around a grid of dots. An essential feature is the rotational symmetry of the design — four-fold for squares and two-fold for rectangles. Any set of four integers ($a\ b\ c\ d$) with the property $c = a + b$ and $d = c + b$ can be the basis for kolams based on the Fibonacci series. Symmetry properties of these kolams arise naturally from the recursive property of the Fibonacci series. Hence, the Fibonacci recurrence is an elegant mathematical tool for design of kolams. A figure shows how this recurrence can be used for construction of square and rectangular kolams of different sizes. As examples, kolams with Fibonacci quartets ($1\ 2\ 3\ 5$), ($2\ 3\ 5\ 8$) and ($5\ 8\ 13\ 21$) are shown. A desirable feature of kolams is that they are 'single-loop'. Since each Fibonacci kolam has at least five modules, a single loop can only be achieved with specific splices between modules. Using probability theory and matrices, we examine the relationship between the numbers of splices and loops. Results show that regardless of the initial number of splices, the end result is one, three or five loops. Fibonacci kolams can be drawn on the computer using eight basic shapes derived from 8 equations. Such computer generated kolams are attractively usable as a 14 x 14 board game.

Every morning before sunrise, women in the south of India clean their doorsteps and draw elaborate figures with rice powder in front of their homes. Through the day, these drawings get walked on, washed out in the rain, or blown away by the wind. A new figure is drawn on the next day. These drawings are thought to bring prosperity to the homes and they are a sign of welcome. They are known as *kolams* and the mathematics behind this folk art is fascinating.

Special kolams, large with bewildering complexity are drawn on festival days. On such occasions, kolam competitions are held in temples. The folk art is handed down through generations of women from historic times, dating perhaps thousand years or more. Today they are nurtured by housemaids and housewives both in rural and urban areas. A kolam is usually drawn around dots, which are trickled on the floor in a regular grid, as a template.

The designs are intricate and creative, governed by some broad rules. One common rule is four-fold symmetry: viewed from all the four sides North, East, South and West the kolam appears the same. A Kolam may have multiple loops (the large kolam in Figure 1 has three) but a single loop is considered special and is harder to achieve, see wikipedia.org/wiki/Kolam and www.ikolam.com.

The simplest geometrical shape with four-fold symmetry is the square. Rectangles have only two-fold rotational symmetry. They serve as modules in building larger squares. More complex shapes can be broken down into squares and rectangles. Therefore we consider only square and rectangular kolams and we will see how Fibonacci recurrence can be used to generate them.

Photo: Wikimedia Commons, Arulraja

Figure 1 Various Kolams in front of a house in Tamil Nadu (India). The three larger Kolams are of the type considered in this paper. The two small Kolams on the red tiles are of a different style.

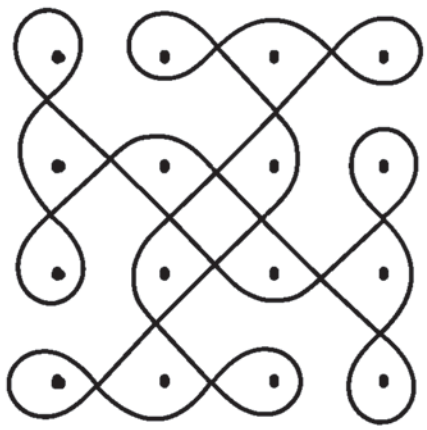


Figure 2 A square kolam with four-fold symmetry.

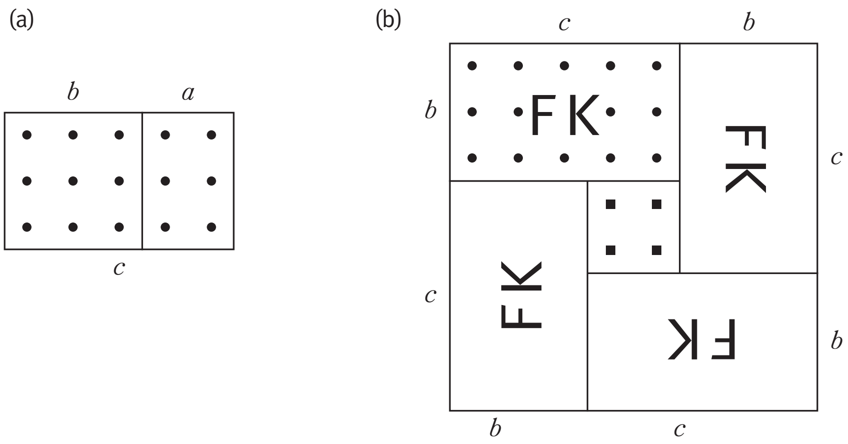


Figure 3 The quartet $(2\,3\,5\,8)$ generates a template for a kolam.

Fibonacci recurrence

In Fibonacci recurrence, a series of integers is generated in which every integer is a sum of two preceding integers. Given two seed numbers all the subsequent integers are determined. The simplest such series starts with seeds 0, 1 and is familiar to all of us:

$$0\;\;1\;\;1\;\;2\;\;3\;\;5\;\;8\;\;13\;\;21\;\;34\;\;55\;\;89\;\;144\ldots \tag{1}$$

These Fibonacci numbers occur in many branches of science. In geometry they are related to pentagons, decagons and the 3-dimensional Platonic solids. Rectangles with sides as consecutive Fibonacci integers are known as *golden rectangles*. The ratio of the sides approaches a limiting value $\phi = (1+\sqrt{5})/2 = 1.61803\ldots$ called the *golden ratio*. This ratio is ubiquitous in nature: in the branching of trees, arrangement of leaves, seeds and flower petals, spiral patterns of florets in sun flowers, spiral shapes of sea-shells et cetera. The golden ratio is also believed to figure prominently in Western art: in architecture (pyramids, Parthenon) paintings (Leonardo da Vinci) sculpture, poetry (Virgil) and music [4]. Many claims are controversial.

The Fibonacci series appeared in the book *Liber Abaci* (1202) by the Italian mathematician Leonardo of Pisa, also known as Fibonacci. It is claimed that the numbers appear in the analysis of prosody of Sanskrit poetry by Acharya Hemachandra in 1150 A.D., 52 years before *Liber Abaci* [8]. For more on Fibonacci numbers see Martin Gardner [2].

For kolam designs we are interested in a set of four consecutive integers in a generalized Fibonacci series such as $(3\,5\,8\,13)$, $(3\,4\,7\,11)$. In a quartet $Q(a\;b\;c\;d)$ we have

$$\begin{aligned} c &= a+b, \\ d &= b+c = a+2b, \end{aligned} \tag{2}$$

and we leave it to the reader to verify that the $a\;b\;c\;d$ are related as follows:

$$bc = b^2 + ab, \tag{3a}$$
$$d^2 = a^2 + 4bc. \tag{3b}$$

Expressed in standard notation equations (3a) and (3b) are

$$F_{n-1}F_n = F_{n-1}^2 + F_{n-1}F_{n-2} \quad (n>1), \tag{4a}$$
$$F_n^2 = F_{n-3}^2 + 4F_{n-2}F_{n-1} \quad (n>3), \tag{4b}$$

for given F_0 and F_1.

Equations (3a) and (3b) are the basis of Fibonacci kolams. They have geometrical counterparts. In Figure 3(a) a square d^2 has a smaller concentric square a^2 and the space between the two is filled up with four rectangles $(b\times c)$ placed in a cyclical pattern. This construction is the essence of Fibonacci kolams. It ensures four-fold symmetry. The rectangles are identical and need have no symmetry property, but the central square a^2 should have four-fold symmetry, as in Figure 3(b). It contains golden rectangles with the sides in ratio $\phi = (1+\sqrt{5})/2$ just as in the case of the Fibonacci series (1). Using equations (3a) and (3b) one can build a hierarchy of square and rectangular kolams leading up to any desired size.

Rangoli

The traditional South Indian kolam is based on a grid of points and is known as PuLLi (dot) kolam or NeLi (curve) kolam in Tamil Nadu. There are also kolams that are free geometric shapes that usually has bright colours. Such drawings are called *rangoli* and they are popular in North India. Special kolams, large with bewildering complexity are drawn on festival days. On such occasions, kolam competitions are held in temples.

A rangoli at Chennai (flickr.com, B. Balaji)

The Zeckendorf array

Edouard Zeckendorf was an army doctor from Liège with broad scientific and artistic interests. In 1947 he was sent to Pakistan as a member of the United Nations peace keeping force. He stayed there for several years. During that time he wrote a paper on the Fibonacci numbers. The Zeckendorf array is named after him. Each next row in the array starts by selecting the first and third from the remaining numbers, and proceeds by Fibonacci recurrence. To construct a square Fibonacci kolam, one can choose a quartet from this array.

1	2	3	5	8	13	21	…
4	7	11	18	29	37	66	…
6	10	16	26	42	68	110	…
9	15	24	39	63	102	165	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Square Fibonacci kolams

Using equations (3a) and (3b) a sequence of square kolams can be built in a hierarchical scheme as shown in the 'skeleton diagrams' in Figure 4. The larger square consists of a smaller square and four cyclic rectangles. Assuming the five constituents are all single loop, our task is to merge them together at splicing points which are symmetrically placed to preserve four-fold symmetry and create a single loop. In general, when all the splices are completed the final outcome will be multiple loops. This is achieved by a careful choice of splicing points.

Some examples of Fibonacci kolams

The construction is based on $Q(1\,2\,3\,5)$ with $5^2 = 1^2 + 4(2\times 3)$. In Figure 5(a) the square 5^2 contains four 2×3 rectangles around a central dot. The rectangles are merged around the central dot in a four-way splice ($\Diamond$) which results in a single loop kolam. In Figure 5(b) there are eight additional splicing points marked in blue and green. This gives us another single loop Kolam. To illustrate the effect of the choice of splicing points on the number of loops, one of the blue splicing points is omitted in Figure 5(c). Here we get a kolam with five loops.

The number of possible 5^2 Fibonacci kolams depends on the number of distinct 2×3 rectangles and the splicing choices. Below we shall see that there are 30 distinct 2×3 rectangles each giving a different 5^2 kolam.

A 3×5 Fibonacci kolam can be obtained by splicing together a 3^2 and a 2×3 rectangle ($3\times 5 = 3^2 + 2\times 3$). In Figure 6(a), the 3^2 has three loops but if it is merged with a 2×3 rectangle at three points the result is a single loop 3×5 with two-fold symmetry as in Figure 6(b).

The 8^2 Fibonacci kolam is based on $Q(2\,3\,5\,8)$: $8^2 = 2^2 + 4(3\times 5)$. A 3×5 Kolam is now available from Figure 6. To get 2^2 can $Q(0\,1\,1\,2)$ be used? Figure 7(a) is the first attempt, but it has an empty unit cell at the centre. The generator is $2^2 = 0^2 + 4(1\times 1)$. In Figure 7(b) the four 1×1 'rectangles' are the four circled dots and the 0^2 can be imagined as a small empty square at the centre with area approaching 0, i.e. the square collapses into a point with dimension 0. The clover-leaf pattern with four leaves touching at the centre is the obvious answer (Figure 7(c)).

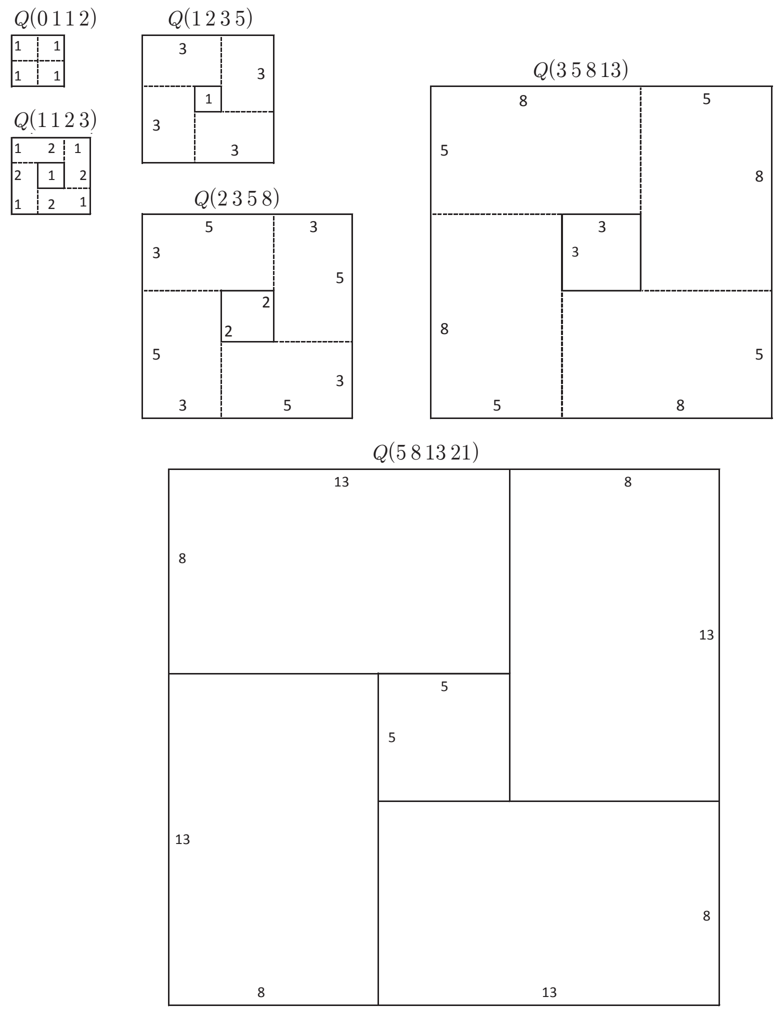


Figure 4 Skeletons of squares for Fibonacci kolams

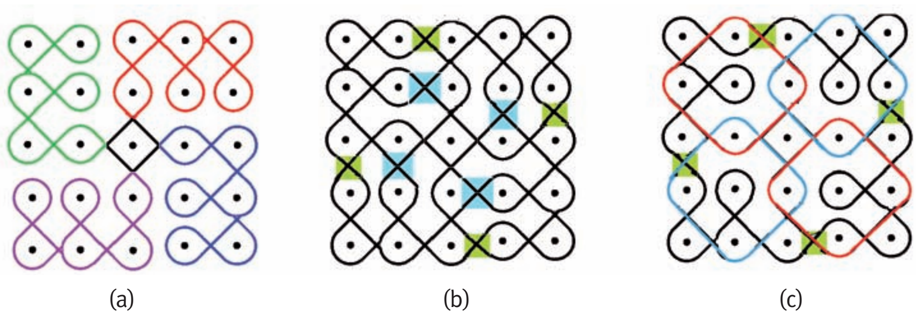


Figure 5 Three 5×5 kolams with a four-fold symmetry. (b) and (c) are created from (a) by splicing and unsplicing.

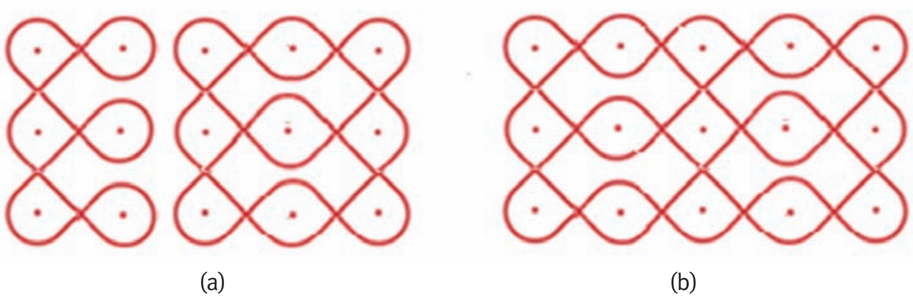


Figure 6 Creating a 3×5 kolam with a single loop from two separate kolams.

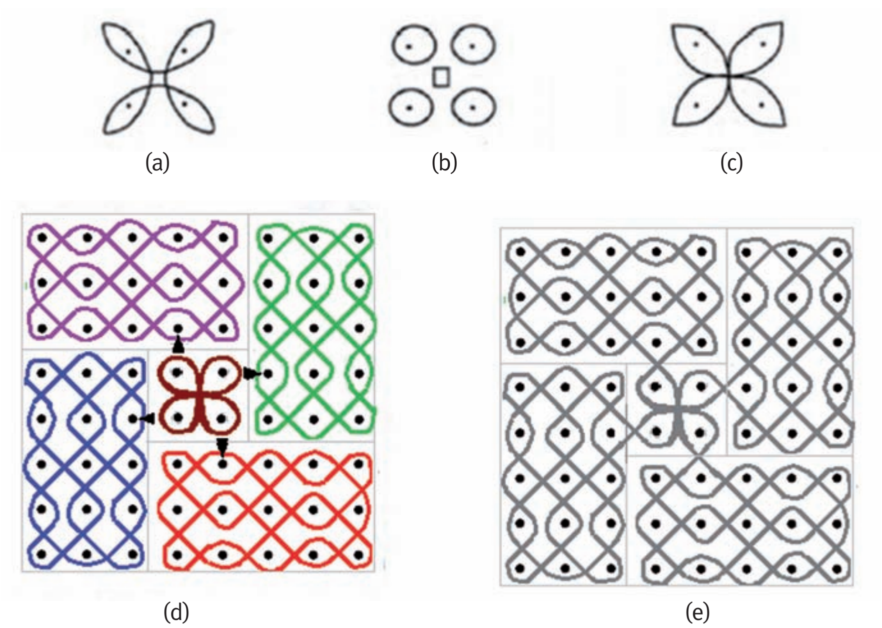


Figure 7 A Fibonacci 8×8 kolam.

This pattern occurs rarely in traditional kolams. Clover-leaf patterns generally occur in square Fibonacci kolams whose sides are even and enhance the overall aesthetics of the kolams.

General construction of Fibonacci kolams
Every Fibonacci kolam — square or rectangle — is composed of a square and rectangles. This is seen in the recursive equations (4a) and (4b). They can be used to generate square and rectangular kolams of any order. The procedure is graphically presented in Figure 8. There are three blocks (I, II, III) each with two columns. The middle block (II) contains the modules used for building the 'square Fibonacci kolams' in the right block (III) and the 'rectangular Fibonacci kolams' in the left block (I). Modules are shown in italics in block II. For instance, to find the composition of a 13^2 Fibonacci kolam (in III) follow the arrows

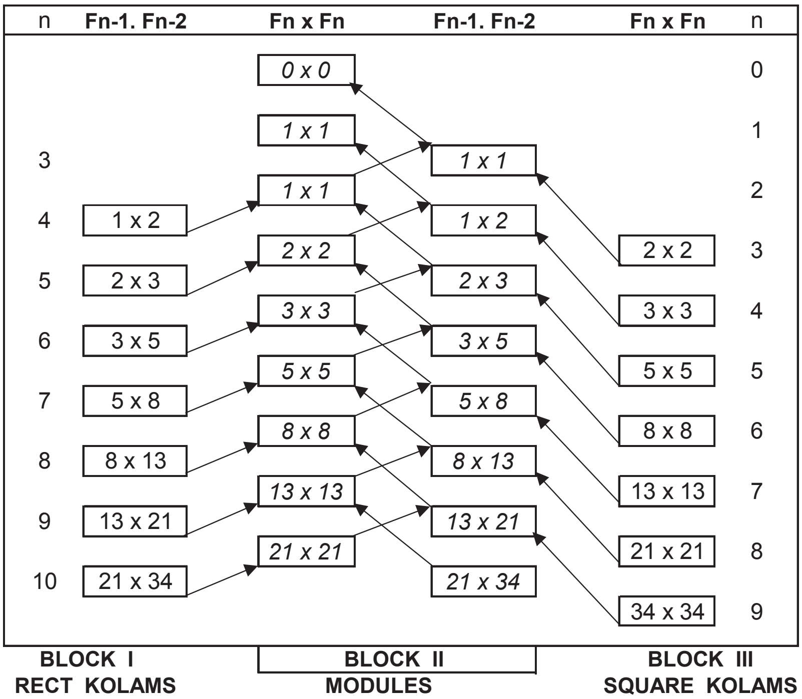


Figure 8 Recursive procedure for construction of square and rectangular Fibonacci kolams.

leading to module II (5×8 and 3^2). Similarly the 8×13 rectangle in block I traces its composition to the module 8^2 and 5×8 in block II. The origins of all square and rectangular kolams can be traced to $F_1^2 = 1^2$ or $F_2^2 = 1^2$.

Every third Fibonacci number F_{3m} ($m = 1,2,3,\ldots$) is even since in $Q(a\ b\ c\ d)$ $d - a = b + c - a = 2b$ is an even number. Therefore d and a are both either even or odd. Since $F_3 = 2$ is even, so are F_6, F_9, $F_{12}, \ldots$ (8, 34, $144, \ldots$). Since $F_1 = F_2 = 1$, all the other Fibonacci numbers are odd.

Figure 9 shows how to build a Fibonacci kolam recursively. The 21^2 Fibonacci kolam is constructed from an arrangement of four 8×13 kolams around a 5^2 kolam. They are spliced together at six sets of points (24 in all). Each 8×13 kolam on its turn is a union of an 8×8 and a 5×8 kolam.

The Fibonacci kolams described so far are all based on the Fibonacci series (1). This is a particular example of a generalized Fibonacci series G_n $(n = 0,1,2,3,\ldots)$ where the first two starting numbers G_0 and G_1 are α and β. The Fibonacci series corresponds to $\alpha = 0$ and $\beta = 1$. If $\alpha = 2$ and $\beta = 1$, we get the series

$$2\ 1\ 3\ 4\ 7\ 11\ 18\ 29\ 47\ 76\ldots,$$

known as *Lucas series*. As long as the Fibonacci recursion

$$G_n = G_{n-1} + G_{n-2} \quad (n > 1)$$

is used, the quartet relation $Q(a\ b\ c\ d)$ with equation (3a) and equation (3b) applies. In other words in equations (4a) and (4b) F_n can be replaced by G_n. This permits generalized Fibonacci kolams of any desired order, i.e. $n \times n$ or $m \times n$ for all m, n. In particular in the Lucas series 4 and all the succeeding numbers are different from Fibonacci numbers. For example a 4×4 generalized Fibonacci kolam can be based on $Q(2\,1\,3\,4)$: $4^2 = 2^2 + 4(1 \times 3)$. This is illustrated in Figures 10(a) and 10(b). A variant of 4^2 based on $Q(0\,2\,2\,4)$: $4^2 = 0^2 + 4(2 \times 2)$ is shown in Figures 10(c)–(f).

How many distinct kolams exist of a given size? This is a problem in combinatorics since kolams are built from smaller sub-units. Starting with the smallest square 2^2 single-loop kolams, the only symmetric version is the 'cloverleaf' pattern in Figure 11(a). The asymmetric versions in Figure 11(b) are used to build 2×3 rectangles (as 2^2 and 1×2). After enumerating all possible 2×3 kolams it

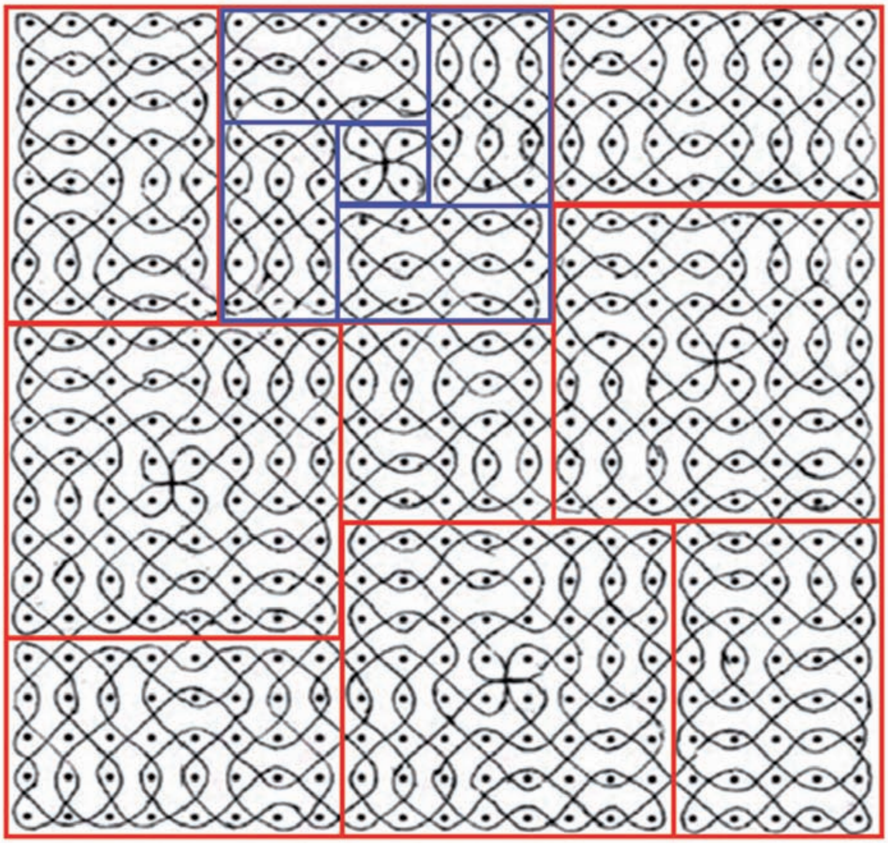


Figure 9 A 21×21 Fibonacci kolam drawn on a skeleton diagram.

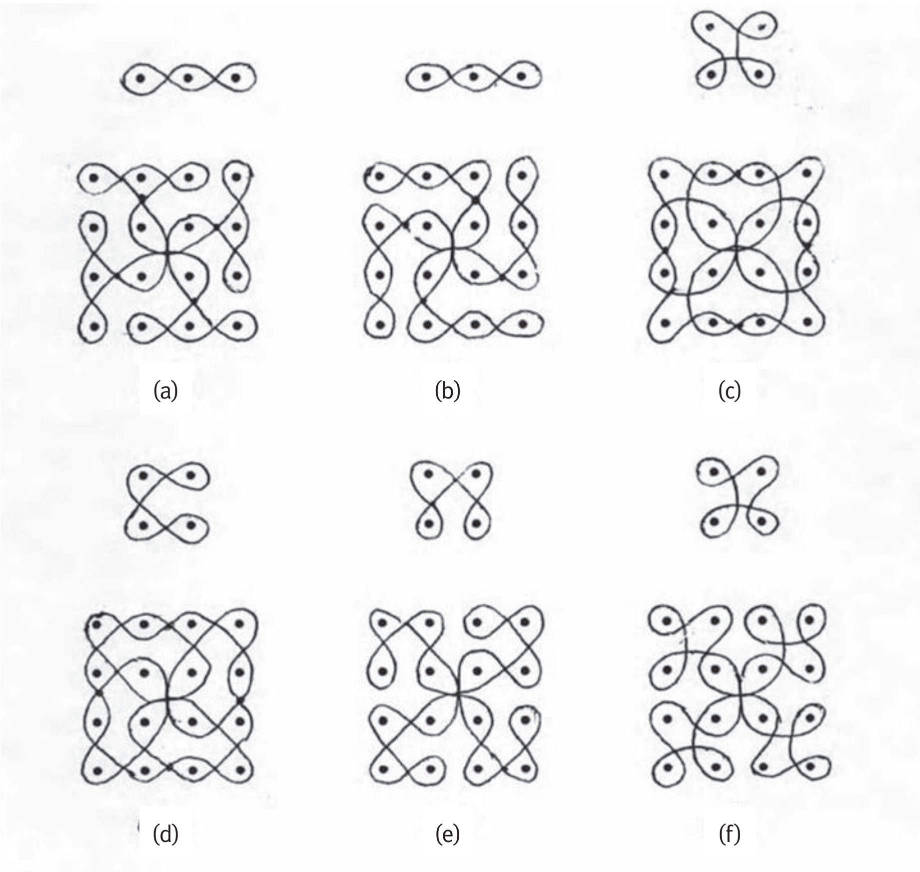


Figure 10 Building 4×4 kolams from 3×1 or from 2×2 kolams.

was found that the six kolams shown in Figure 11(c) are suitable as basic modules. They are labelled E, R, G, H, U, S, so that the alphabet patterns reflect the shape and symmetry property of the kolams. Enumeration of larger kolams is a complex problem.

Symmetry properties

The dihedral group D_4 is the symmetry group of the square. It contains the rotations by 0°, 90°, 180°, 270° which we label by I, $R(90)$, $R(180)$, $R(-90)$. The reflection operators, or mirrors, are $M(X)$, $M(Y)$, $M(45)$ and $M(-45)$ for reflections about X-axis, Y-axis, diagonal $Y = X$ and the anti-diagonal ($Y = -X$). These eight operators

$$I \;\; R(90) \;\; R(180) \;\; R(-90)$$

$$M(X) \;\; M(Y) \;\; M(45) \;\; M(-45)$$

form the dihedral group. How do these operators alter the shape of ERGHUS kolams? In Figure 12, the basic kolam shapes (K) are in the top box. Successive rows show the effect of different operators on K. Some interesting features are as follows:

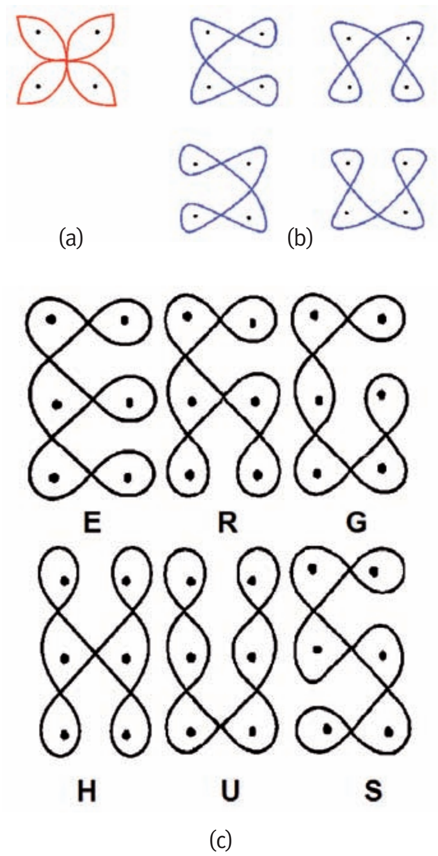


Figure 11 The 2×2 and 2×3 kolams – building blocks of the Fibonacci kolams.

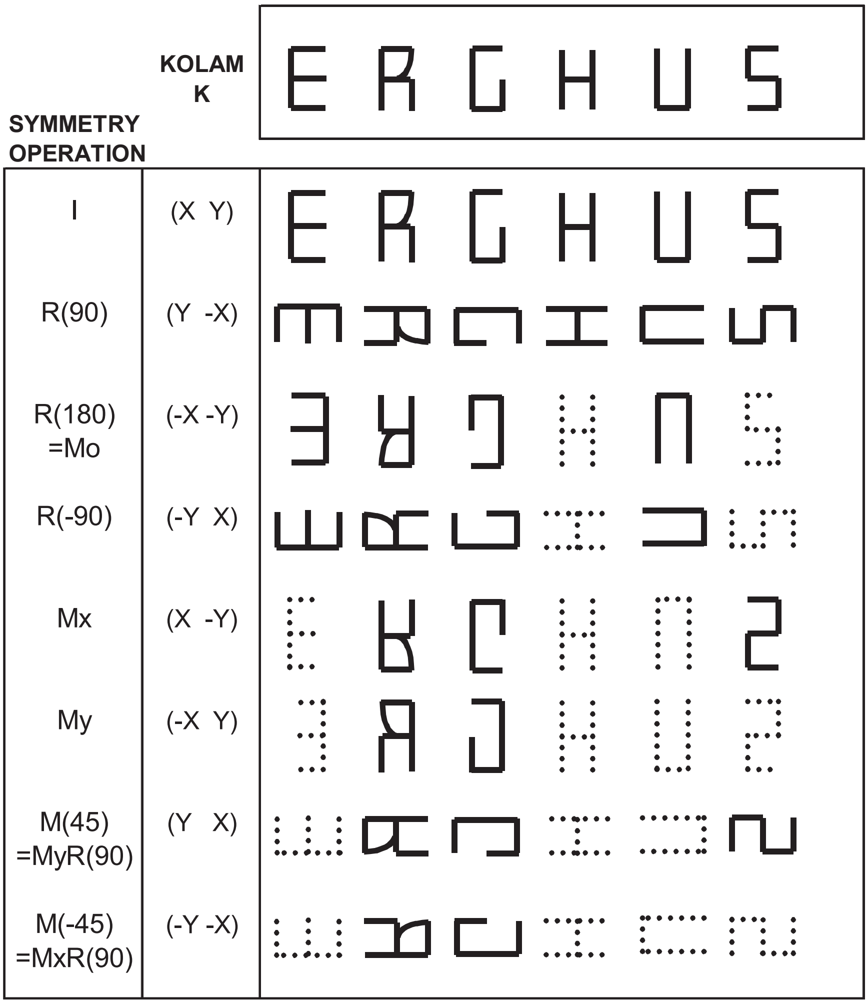
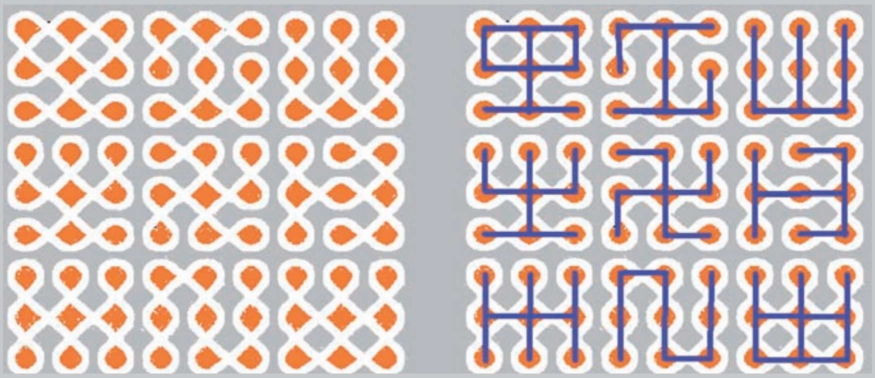


Figure 12 Transformations of the ERGHUS kolams.

Nine goddesses

The Hindu festival of Navaratri, which means *nine nights* in Sanskrit, celebrates the nine incarnations of Shakti. There is a different kolam for each incarnation, all single loops around nine dots. Each kolam has a symmetry. Seven kolams have a single symmetry, a mirror or a point reflection, but two have more. The central kolam has a four-fold symmetry. It is a swastika, which means *good fortune* in Sanskrit. It is not hard to find other single loops around nine dots. However, it is hard to find another single loop which has a symmetry.



Nine kolams for nine incarnations on the left. Graphs display the symmetries of the kolams on the right.

1. Of the total 48 (6×8) patterns only 30 are distinct. The remaining 18 are repetitions shown as dotted patterns.
2. Only for R and G all the eight operators result in distinct patterns (columns 2 and 3). For E, U, S only four are distinct. For H only two give distinct patterns [I and $R(90)$].
3. Symmetric kolams often have a special meaning, as described in the text box on the nine goddesses of Navaratri.

Other Fibonacci kolams

In generating square kolams invariably rectangular kolams appear as constituents. These rectangles are golden rectangles (ratio of sides $= \phi = 1.618\ldots$), but they lack the two-fold rotational symmetry (appearing the same viewed from north or south). 'Rectangles' with two-fold symmetry can be built with two Fibonacci quartets. In analogy with equation (2),

$$Q_1(a_1\, b_1\, c_1\, d_1), \qquad Q_2(a_2\, b_2\, c_2\, d_2),$$

$$c_1 = a_1 + b_1, \quad d_1 = c_1 + b_1 = a_1 + 2b_1, \tag{5a}$$

$$c_2 = a_2 + b_2, \quad d_2 = c_2 + b_2 = a_2 + 2b_2, \tag{5b}$$

An identity relating all the numbers, in analogy with equation (3b) is

$$d_1 d_2 = a_1 a_2 + 2b_1 c_2 + 2c_1 b_2. \tag{6}$$

The rectangle of sides $d_1 d_2$ is composed of a concentric rectangle $a_1 a_2$ and two pairs of rectangles $b_1 c_2$ and $c_2 b_1$. In each pair, one is rotated with respect to the other to ensure two-fold rotational symmetry. In addition, if rectangle $a_1 a_2$ has two-fold symmetry, the overall kolam $d_1 d_2$ has two-fold symmetry. The construction is illustrated in a 6×7 kolam in Figure 13 using

$$Q_1(4\,1\,5\,6), \qquad Q_2(1\,3\,4\,7).$$

This 6×7 kolam is named *Kaprekar kolam* [5,6] as all the four digits of the Kaprekar constant 6174 appear in the quartets $Q_1(4\,1\,5\,6)$ and $Q_2(1\,3\,4\,7)$.

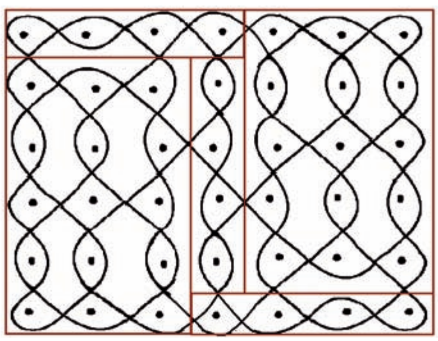


Figure 13 Kaprekar kolam drawn on its skeleton diagram.

Kaprekar's constant
Take any four digits, not all equal. Arrange them in descending and ascending order. Subtract to get four new digits. Repeat. D.R. Kaprekar, an Indian school teacher who published a great number of recreational math papers, discovered that this process always ends up at 6174. This is now called Kaprekar's constant.

$$8765 - 5678 = 3087$$
$$8730 - 0378 = 8352$$
$$8532 - 2358 = 6174$$
$$7641 - 1467 = 6174$$

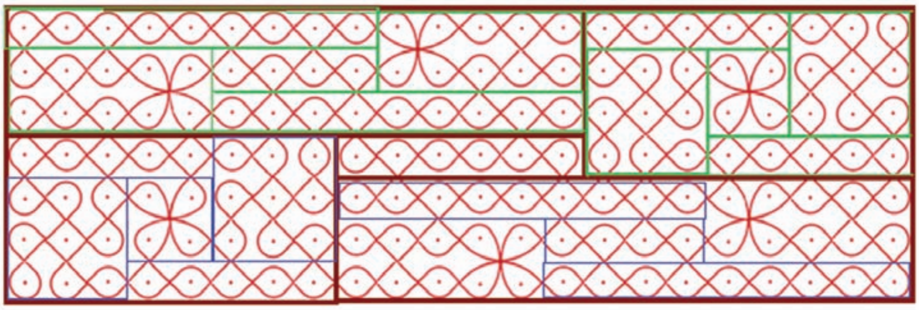


Figure 14 The π-kolam drawn on its skeleton diagram.

Rectangular kolams with two-fold symmetry can be drawn with any desired d_1 and d_2. A rectangle of sides 7×22 has the ratio $22/7 = 3.1428\ldots$ which is a very good approximation to π. A rectangle with sides 7×19 has the ratio $19/7 = 2.7412\ldots$ which is a close approximation to e, the natural base for logarithms. Both π and e along with the golden ratio ϕ, are among the most important mathematical constants.

The π-kolam is coded by the quartets

$$Q_1(1\,3\,4\,7),$$
$$Q_2(6\,8\,14\,22).$$

At the centre is the linear string 1×6 and the enveloping rectangles are 3×14 and 4×8 (in pairs). They serve as modules for the 7×22 kolam in Figure 14.

In Figure 15 is drawn a 9×13 kolam based on the quartets

$$Q_1(5\,2\,7\,9),$$
$$Q_2(7\,3\,10\,13).$$

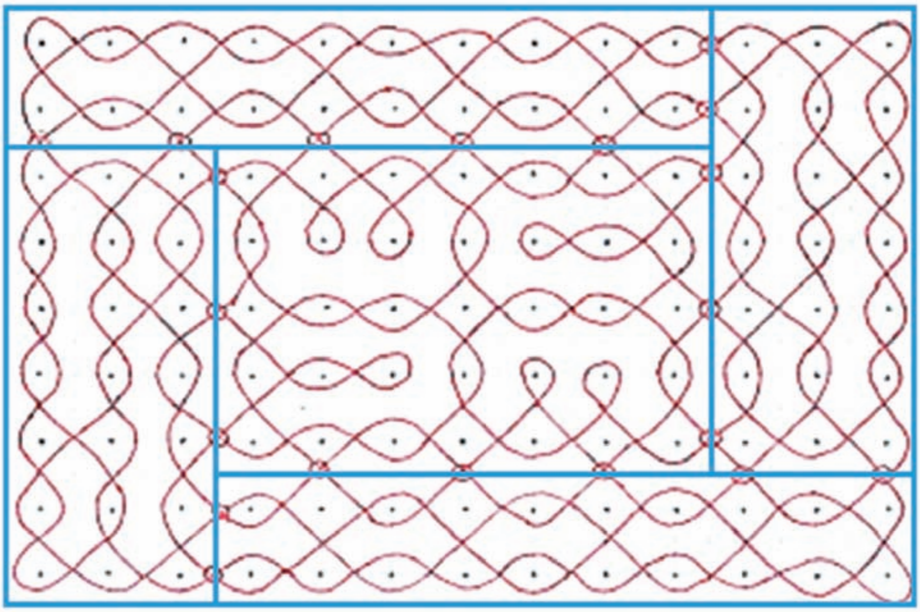


Figure 15 A single loop 9×13 kolam.

The constituent rectangles 5×7 (at the centre) and the surrounding rectangles 2×10 and 7×3 are spliced together at 2×10 points to form a 9×13 single-loop kolam.

If we remove the central 5×7 rectangle from this kolam, then we get the 'window-frame' kolam in Figure 16. A careful inspection reveals that it has two loops. It can be shown that a window-frame kolam can never be single loop whatever be the choices of quartets and splices. This is due to the fact that there is no central rectangle. Since the starting configuration has four loops, an even number, parity conservation dictates the minimum number of loops as two. We consider the number of loops in the next section.

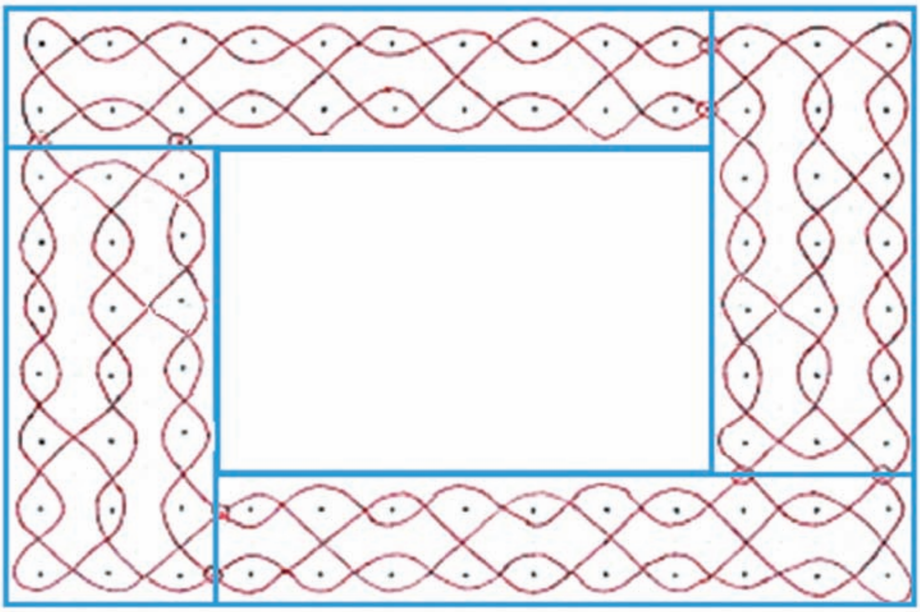


Figure 16 A window frame kolam with two loops.

Evolution of loops

A Fibonacci kolam is assembled by merging or splicing five smaller modules at a set of points. In general, the final kolam has multiple loops. A single loop is achieved by a suitable choice of splices. Another requirement prompted by aesthetic reasons is that 'four-sided islands' are not allowed (see section on 'Square Fibonacci kolams'). To elaborate: islands are empty bounded regions without a dot. A four-sided island appears like a diamond without a dot (◊). Four-sided islands arise mainly when splices are made at adjacent points. So generally splices are made at alternate points. They occur along the edges of the constituent rectangles. In practice, a good strategy to obtain maximal splicing consistent with a single loop is the following. Splice all allowed splicing points at one 'go'. If the number of loops is one, the task is done. If not, unsplice one or more sets of four points to make the kolam single loop; the choice of the sets will require some 'trial and error' experimentation.

Splicing and unsplicing rules

From a detailed analysis of loop evolutions the following empirical rules emerge.

1. A splice between two loops gives one loop.
2. A splice within a single loop splits it into two loops.

As a converse to above, the unsplicing rules are the following.

1. Unsplicing at the intersection of two loops gives one loop.
2. Unsplicing within a single loop will split it into two loops.

If there are $l(>1)$ loops, an operation (splice or unsplice) results in $l-1$ or $l+1$ loops, a binary option. A set of four splices can be modelled as a binary tree with four nodes.

Loop evolution as a binary tree

In Figure 17 a 'top-down' tree has its root at the top which stands for 1 (number of loops). After the first splice there are two loops. The second splice alters the number of loops to one or three at the next level depending on whether the splice is in a single loop or between two loops. We assume this is random and assign equal probabilities ($\frac{1}{2}$) to the two branches. If the third splice occurs in a single loop the result is two loops with probability one. If it occurs in one of the three loops the outcome is two or four loops with probabilities $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Among l loops if a random splicing point is chosen between two loops, the loops are likely to be the same loop with probability $\frac{1}{l}$ and different with probability $\frac{l-1}{l}$. Hence the branches $3 \to 2$ and $3 \to 4$ are assigned probabilities $\frac{2}{3}$ and $\frac{1}{3}$. After the fourth and last splice the end points are 1, 3 or 5. Their relative probabilities are calculated by multiplying the probabilities assigned to each evolutionary path. For example $l=1$ is the end point of two pathways $1\text{-}2\text{-}1\text{-}2\text{-}1$ or $1\text{-}2\text{-}3\text{-}2\text{-}1$. Their respective probabilities are $1\times\frac{1}{2}\times1\times\frac{1}{2}=\frac{1}{4}$ and $1\times\frac{1}{2}\times\frac{2}{3}\times\frac{1}{2}=\frac{1}{6}$. Together they add up to $\frac{5}{12}\,(\approx 0.417)$. A similar calculation for $l=3$ as the end point shows three pathways $1\text{-}2\text{-}1\text{-}2\text{-}3$, $1\text{-}2\text{-}3\text{-}2\text{-}3$ and $1\text{-}2\text{-}3\text{-}4\text{-}3$ with probabilities $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ adding up to $\frac{13}{24}\,(\approx 0.542)$. Finally the probability of $l=5$ is $1\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}=\frac{1}{24}\,(\approx 0.041)$. Summarising, starting from a single loop, after a set of four splices the numbers of loops is 1, 3 or 5 with probabilities 0.417, 0.542 and 0.041, respectively.

Notice that a Fibonacci kolam is built from five modules and therefore we need to splice five loops. We leave that three to the reader. The evolution of three splices is shown in Figure 18. As for $i=1$, here too the branches are labelled with probabilities. Starting with $i=3$, the probability of final $j=7$ is $\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}=\frac{1}{360}\;(\approx 0.0028)$ less than 0.3%. How do the loops evolve further when another four-splice is added? Ultimately we have to deal with a large number of four-splices. This can be handled elegantly using matrices of loop probabilities.

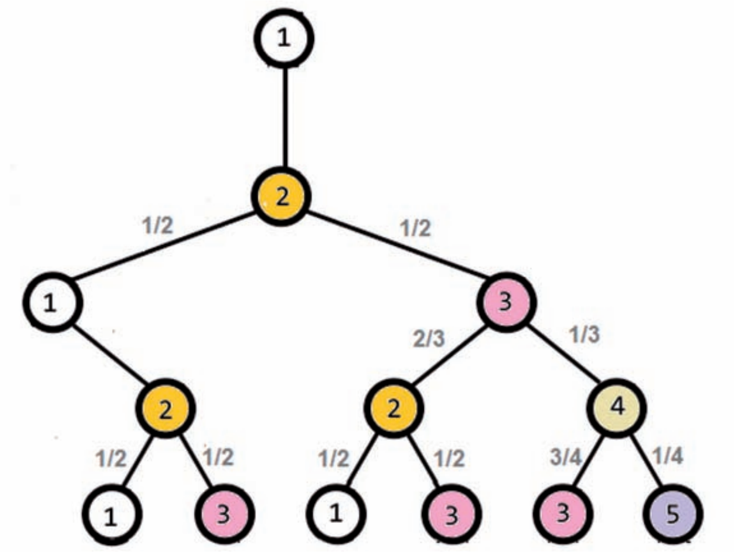


Figure 17 The binary tree associated to splicing a single loop kolam four times, including the transition probabilities. Four-fold symmetry requires four splices.

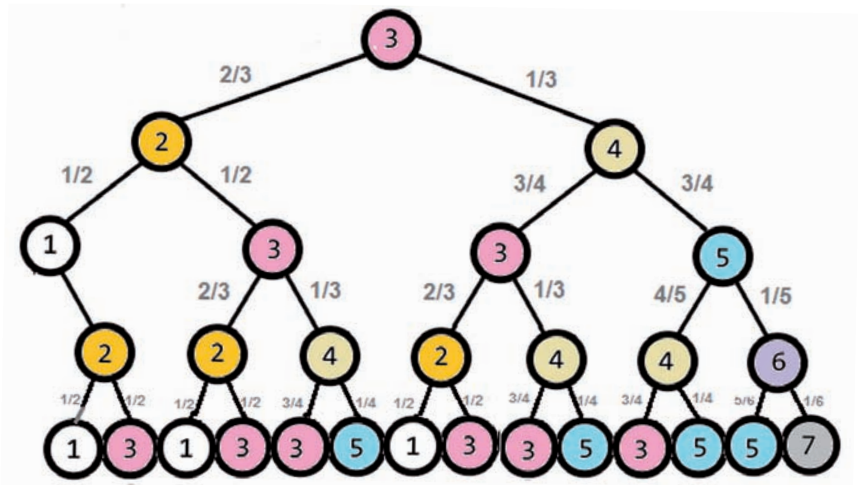
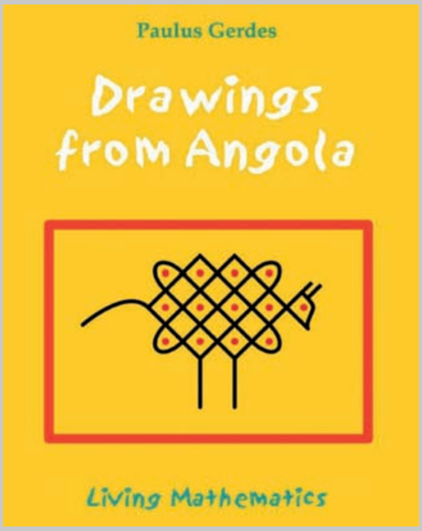


Figure 18 The binary tree for splicing three loops illustrates the bias towards a small number of loops.

Curvy loops around the globe

Drawing loops around a dotted grid is a common practice in several places around the globe. In South-West Africa, the Chokwe and Luchazi people draw curves called *lusona* in the sand. These curves can depict fables or riddles or animals. Similar curves can be found on the Vanuata Islands (New Hebrides) in the Pacific Ocean. Paulus Gerdes, a Dutch mathematician who lived and worked in Mozambique, wrote several books and papers about them. Scientific works such as [3], but also children's books. A kolam or a lusona is drawn by simple hand movements, turning left and right. It is natural to study the movements from an algorithmic point of view. This was done by Gabrielle Allouche, Jean-Paul Allouche and Jeffrey Shallit. They constructed a kolam from the Thue–Morse sequence in [1].



The cover of a children's book by Paulus Gerdes

Loop probability matrix.

Let $p(ij)$ be the probability of initial i loops leading to j loops for $i,j = 1,3,5$ as in Figure 17. The probabilities define a 3×3 matrix S of nine elements

$$\begin{matrix} p(11) & p(13) & p(15) \\ p(31) & p(33) & p(35) \\ p(51) & p(53) & p(55) \end{matrix}$$

We have already determined $p(11) = 0.417$, $p(13) = 0.542$, $p(15) = 0.041$. The full matrix is

$i\backslash j$	1	3	5^*
1	0.417	0.542	0.041
3	0.361	0.557	0.082
5	0.200	0.570	0.230

*These probabilities include small contributions from $j = 7,9$.

How do the loop probabilities change after two sets of splices? Let the new probabilities be $q(ij)$. Then

$$q(ij) = p(i1)p(1j) + p(i3)p(3j) + p(i5)p(5j) \quad \text{for } i = 1,3,5. \qquad (7)$$

Equation (7) is exactly the same as the rule for multiplication of matrix S by itself. Matrix $|q(ij)|$ is simply $S * S\,(= S2)$, where $*$ denotes matrix multiplication. To find the probability after three sets of four-splices multiply $S2$ by S to get $S3$, etcetera. These matrices are presented in Figure 19. The matrix elements quickly converge to nearly constant values.

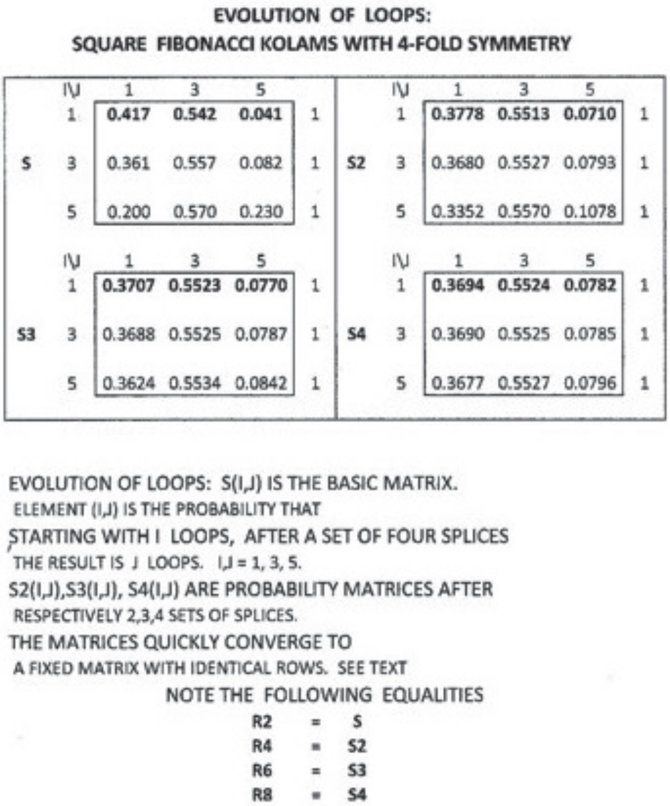
$$p(11) = 0.37, \quad p(13) = 0.55, \quad p(15) = 0.08.$$

$p(ij)$ is the limiting value of probability after a large number of four-splices. Notice all the rows are nearly the same. The probability of loops after a large number of four-splices is independent of the starting number. It is the stationary distribution of this Markov chain.

The most probable number of loops at the end-point is three with a probability of 0.55 and the probability of the desired single loop is 0.37. This leaves a small probability of 0.08 for the probability of five or more loops. The relative probabilities for $j = 1$ and 3 are roughly $2:3$. These results are consistent with an empirical observation that three loops is the most likely end result even for large kolams with a large number of splices.

One can estimate the maximum number of splicing points available in constructing a Fibonacci kolam. Splices occur along edges of the rectangles shown in Figure 3. Since splices are chosen at alternate points along the edge, the maximum number of splices along an edge is $\approx c/2$ for square kolams. Since four-fold symmetry forces splicing at four symmetrically placed points, the maximum, number of splices is $\approx 2c$. For rectangular kolams the maximum number of splices $\approx c_1 + c_2$. In large kolams of size $d^2\,(d = 20\text{–}30)$, $c \approx 0.6d$ and the maximum number of splices is $1.2d$ or $24\text{–}36$. The number of four-splices is $6\text{–}9$.

It is a remarkable fact that even with ≈ 30 splices the final number of loops j is most likely 1, 3 or 5 with relative probabilities 0.37, 0.55 and 0.08. This is because


EVOLUTION OF LOOPS:
SQUARE FIBONACCI KOLAMS WITH 4-FOLD SYMMETRY

	I\J	1	3	5	
S	1	0.417	0.542	0.041	1
	3	0.361	0.557	0.082	1
	5	0.200	0.570	0.230	1

	I\J	1	3	5	
S2	1	0.3778	0.5513	0.0710	1
	3	0.3680	0.5527	0.0793	1
	5	0.3352	0.5570	0.1078	1

	I\J	1	3	5	
S3	1	0.3707	0.5523	0.0770	1
	3	0.3688	0.5525	0.0787	1
	5	0.3624	0.5534	0.0842	1

	I\J	1	3	5	
S4	1	0.3694	0.5524	0.0782	1
	3	0.3690	0.5525	0.0785	1
	5	0.3677	0.5527	0.0796	1

EVOLUTION OF LOOPS: S(I,J) IS THE BASIC MATRIX.
ELEMENT (I,J) IS THE PROBABILITY THAT
STARTING WITH I LOOPS, AFTER A SET OF FOUR SPLICES
THE RESULT IS J LOOPS. I,J = 1, 3, 5.
S2(I,J),S3(I,J), S4(I,J) ARE PROBABILITY MATRICES AFTER
RESPECTIVELY 2,3,4 SETS OF SPLICES.
THE MATRICES QUICKLY CONVERGE TO
A FIXED MATRIX WITH IDENTICAL ROWS. SEE TEXT
NOTE THE FOLLOWING EQUALITIES
R2 = S
R4 = S2
R6 = S3
R8 = S4

Figure 19 Convergence of the transition matrices towards the stationary distribution.

the pathways of loop evolution tend to alternately rise and fall keeping the final j to low values as illustrated by the binary tree diagrams Figures 17 and 18. What goes up must come down!

The basic principles underlying the Fibonacci kolams are well understood and the rules for splicing/unsplicing the parts into a whole are simple. More information can be found in [7].

Concluding remarks and summary

Kolams are artistic geometrical drawings with curves and loops around dots in a square grid. Fibonacci Recurrence is used to create a new family of kolams with ground rules — single loop, symmetry (four-fold for square, and two-fold for rectangle) and avoidance of four-sided islands (previous section) — dictated by aesthetics. Using generalized Fibonacci series, kolams of any desired size are possible.

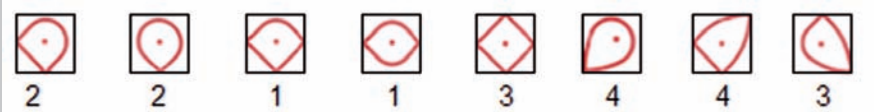
Although squares and rectangles are described in separate sections, in practice they are inseparable. Square kolams as well as rectangular kolams are composed of smaller squares and rectangles as implied by equations (3a) and (3b). The rectangles in section 'Square Fibonacci kolams' are not two-fold symmetric since they are based on only one quarter $Q(a\,b\,c\,d)$. Rectangles in section 'Other Fibonacci kolams' are two-fold symmetric since they are based on two quartets. Hierarchical structures involving squares and rectangles are possible in building kolams of any arbitrary size. The basic modules 2^2, 3^2, 2×3 are based on quartets $Q(0\,1\,1\,2)$ and $Q(1\,1\,2\,3)$. The 2^2, the clover-leaf pattern is unique among kolam patterns. Enumeration of the number of these basic modules invokes symmetry operators of group theory.

Merging five modules of a Fibonacci kolam at splicing points along their edges to produce a single loop, is governed by simple rules. As the number of splices increases the number of loops goes up and down resulting in a small number of loops 1, 3 or 5 at the end. This can be shown as a result of the fact that as each splice is added the number of loops is increased or decreased by one. This leads to modelling the evolution of loops as a binary tree and expressing the loop probabilities as elements of a 3×3 matrix (previous section). The relative probabilities of the number of loops 1, 3, 5 converge to limiting values of 0.37, 0.55 and 0.08, respectively, as the number of splices becomes large. After each set of four-splices the number of loops changes by 0, 2 or 4, so that the parity of initial and final number of loops is conserved. Since the starting number of loops is odd (5) the final number of loops is also odd (1, 3 or 5). When the starting number is even (4), as in the case of window-frame *kolam*, the final number of loops is 2 since the parity has to be even.

It is remarkable that all Fibonacci kolams can be assembled with only eight basic shapes (see box above). Rotations and reflections give 31 distinct shapes. This enables designing a board game for assembling kolam designs on a board with a square grid. It can be played by one or more players. The game has been adapted for play on a computer. Feedback from more than 800 users has been positive. The game is also physically realised with a 14×14 square grid made of thick cardboard and 300 white tiles with kolam shapes etched on them.

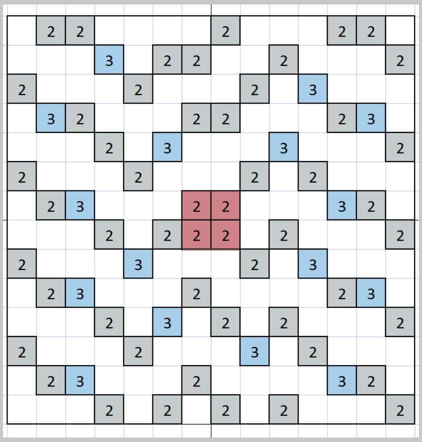
The Kolam Game

Kolams are built from single cells with a single dot encircled by a loop that touches one or more sides. There are eight basic cell patterns. All tiles have a mirror symmetry, except for the last one.



The 8 cells as tiles for a board game.

The *Kolam Game* is played with ten tiles, the eight basic tiles and the mirror of the unsymmetric tile and a blank. Each tile counts for a number of points, as shown below the tiles, with no points for the blank. Players draw tiles from the bank and place them on the board. Tiles have to be contiguous. The game is played on a board, shown on the right. The initial tile has to be placed on one of the central red squares. As in the game of *Scrabble*, there are special squares which multiply the value of the tile. By now, this game has been tested by more than 800 users. For details on the rules of the game, see [7].



The board of the game (14 × 14).

Acknowledgment

This paper is mainly based on a series of papers on Fibonacci Kolams on the website of the author, see vindhiya.com/snaranan/fk [7].

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