KOLAM DESIGNS BASED ON FIBONACCI NUMBERS

Part II: Square and Rectangular designs of arbitrary size based on Generalized Fibonacci Numbers

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In Part I of this article we presented a scheme for creating a class of *kolams* based on Fibonacci numbers. Several square and rectangular *kolams* were displayed. In this Part II, the scheme is generalized to arbitrary sizes using Generalized Fibonacci numbers. The problem of enumeration – the number of possible Fibonacci *kolams* of a given size – is discussed. For 2 x 3 *kolams* symmetry operators forming a group are used to classify them. Finally the scheme is further extended beyond square grids to cover diamond-shaped grids. Possible connections of *kolams* to Knot theory and Group theory are indicated.

1. Introduction.

Kolams are decorative patterns drawn as curved lines around dots in a rectangular grid. In Part I [1] we described a class of *Kolam* designs based on Fibonacci numbers (F_n) in the Fibonacci series

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \quad 144 \quad \ldots\ldots\ldots \qquad (*)$$

They are generated by the simple recursive equation

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad (n > 1) \qquad (1)$$

We constructed Fibonacci Square (FS) *kolams* of size F_n^2 and Fibonacci Rectangular (FR) *kolams* of sides equal to successive Fibonacci numbers F_{n-1} and F_{n-2}. We adopted a few ground rules: square lattices, symmetry (four-fold rotational), single (endless) loops and no unit cells without dots at the centre. These rules enhance the aesthetic value of the designs and help pose mathematical problems for exploration. We presented FS *kolams* of size 3^2 , 5^2 , 8^2 , 13^2 , 21^2 and FR *kolams* of size 2 x 3, 3 x 5, 5 x 8 and 8 x 13 in [1].

The construction is modular: building bigger Fibonacci *kolams* (FKs) from smaller sub-units. The procedure exploits an important property of Fibonacci numbers. Let Q = (a b c d) be a quartet of consecutive Fibonacci numbers. Then

$$b\,c = b^2 + a\,b \tag{2a}$$

$$d^2 = a^2 + 4b\,c \tag{2b}$$

Both are easily proved using $d = c + b,\ \ c = b + a$. The geometrical versions of equations (2a) and (2b) are given in *Figure* 1, which bears duplication from Part I (*Figure* 2) since this is the basis for building FKs. For example for Q = (3 5 8 13) an FS *kolam* 13^2 (d^2) is made up of a smaller FS *kolam* 3^2 (a^2) at the centre and four FR *kolams* of 5 x 8 (b x c) arranged cyclically around the square (*Figure* 7c in Part I). The merit of this construction lies in the fact that the cyclic arrangement automatically ensures four-fold rotational symmetry of the square *kolam* if the central square *kolam* is symmetric.

At the end of Part I, it was mentioned that equations (2a) and (2b) apply also for Generalized Fibonacci numbers defined by

$$G_0 = \alpha,\ G_1 = \beta,\ G_n = G_{n-1} + G_{n-2} \qquad (n > 1) \tag{3}$$

The Fibonacci series (*) is a special case of equation (3) with (α, β) = (0, 1). For example for (α, β) = (2, 1), we have the Lucas series

2 1 3 4 7 11 18 29 47 76 ....

Any quartet Q = (a b c d) of Lucas numbers also satisfies equations (2a) and (2b).

First we will generate “Lucas Square Kolams” of size 4^2 based on Q = (2 1 3 4). As a variant for 4^2 we consider Q = (0 2 2 4) corresponding to (α, β) = (0, 2), and find new patterns (section 2). This is the first step towards producing GFKs of any arbitrary size d^2 by suitably choosing the quartet Q. *Thus the class of GFKs covers square kolams of all sizes* (section 3).

So far we have confined to square cells in square and rectangular grids. Is it possible to extend GFKs to other grids, such as the diamond-shaped grids of the type in *Figures* 1(g), 1(h), 1(i) in Part I? It can be done and we will indicate how to transform a square grid FK to a diamond grid *kolam*.

We have a very broad canvas to create new *kolam* designs and to realise the full potential we need the aid of computer with suitable software of algorithms and graphic aids.

2. Lucas Kolam designs of 4^2 and 7^2.

The Lucas quartet Q = (2 1 3 4): $4^2 = 2^2 + 4(1 \times 3)$ is the basis for a 4^2 square Lucas *kolam*. In *Figures* 2(a) and 2(b) there are four 1 x 3 ‘rectangles’ cyclically placed around a square 2^2 cloverleaf (Part I, *Figure* 5c). The splicing points between the ‘rectangles’ (actually linear strings) and the central square are shown as dark dots. The two figures have different splicing points. Note that the splices come in sets of four, for preserving symmetry (short for four-fold rotational symmetry). These two are the only possible single loop 4^2 Lucas *kolams*. The other designs in *Figure* 2 are explained in section 3.

We can build a 7^2 Lucas *kolam* from Q = (1 3 4 7): $7^2 = 1^2 + 4(3 \times 4)$. The 3 x 4 rectangle can be composed from a 3^2 square and a 1 x 3 linear string. (*Figure* 3a). Four such rectangles can be merged with the centre dot (1^2) in a four-way splice (*Figure* 3b). Note that there is a wide choice for the 3^2 (section 5) since it need not be symmetric; but we have chosen the unique 3^2 square.

3. Generalized Fibonacci *Kolams*: 4^2, 6^2, 9^2 and 10^2.

So far we have designed square *kolams* n^2: n = 2, 3, 5, 8 (Fibonacci) and n = 4, 7 (Lucas). To fill the gap, n = 6, 9 , 10 we need to consider Generalized Fibonacci numbers and we incidentally generate a GF *kolam* of size 4^2 in addition. The quartets corresponding to the *kolams* are as follows:

Kolam	**Q = (*a b c d*)**	**Composition**	**(α, β)**
4^2	(0 2 2 4)	$4^2 = 0^2 + 4\ (2 \times 2)$	(0, 2)
6^2	(0 3 3 6)	$6^2 = 0^2 + 4\ (3 \times 3)$	(0, 3)
9^2	(1 4 5 9)	$9^2 = 1^2 + 4\ (4 \times 5)$	(1, 4)
10^2	(0 5 5 10)	$10^2 = 0^2 + 4\ (5 \times 5)$	(0, 5)

4^2 *kolam*: The rectangles (*b x c*) degenerate into 2 x 2 squares. These squares need not be symmetric since the symmetry requirement is only for the big square (d^2). In *Figure* 2(c), (d), (e), (f) are shown the 2 x 2 modules and the *kolams* they generate. The squares are merged at the (imaginary) centre (0^2 !) in a four-way splice. Additional splices are shown as four dark dots. Note the 2 x 2 squares in *Figure* 2(c) and 2(f) are mirror images (reflected in a vertical axis), yet they yield very different patterns. Also *Figures* 2(d), 2(e) have as their generators 2 x 2 squares which are cyclic (one rotated 90^o).

Figure 2(c) is a beautiful popular *kolam* known as *Brahmamudi* (Brahma's knot) and is used as a basic module to draw a large variety of designs in different grid shapes. *Figure* 2(d) is a variant of *Brahmamudi* with skewed circular loops touching at the centre – an interesting twist inspired by the modular approach.

6^2 *Kolam*: The degenerate 'rectangles' or asymmetric squares 3 x 3 have great variety (section 5) and we have chosen just two as generators for the 6^2 *kolams* in *Figures* 4(a), 4(b). As in the case of 4^2 *kolams*, here too there is a four-way splice at the centre and an additional set of four splices marked by dark dots.

9^2 *Kolam*: This is made up of four 4 x 5 rectangles around a dot (1^2). The rectangle is composed of a 4^2 *Brahmamudi* and linear string of length 4 (1 x 4) spliced to it on the right at two points (*Figure* 3c). Four such rectangles are merged with the central dot in a four-way splice and there are in addition two sets of splices (*Figure* 3d). The result is an intricate complex pattern based on the *Brahmamudi.*

10^2 *Kolam*: Here again the generators are squares as in 4^2 and 6^2 *kolams*. We will choose a 5 x 5 asymmetric *kolam* as the generator. First we describe a general procedure for producing asymmetric square GFK. (The symmetric versions are created as in *Figure* 1b). In the quartet Q = (*a b c d*)

$$d^2 = d\,(b + c) = d\,c + d\,b = d\,c + (c + b)\,b = d\,c + c\,b + b^2$$

d^2 is composed of rectangles *d x c, c x b* and a square b^2 (*Figure* 4 c). The three modules can be suitably joined together to yield a d^2 asymmetric *kolam*. Note that the square *kolam* need not be single-loop, as the requirement of single loop is only for the larger 10^2 kolam. To emphasize this point, we have chosen an asymmetric 5^2 with *three* splices and *two* loops (*Figure* 4d). Four such squares are joined at the centre in a four-way splice (as in 4^2 and 6^2 *kolams*) with two added sets of splices (8 in all). The result is a single loop 10^2 *kolam* (*Figure* 4e).

4. Generalized Fibonacci *Kolams* of Arbitrary Sizes.

After dealing with particular cases of FS *kolams* of size n^2 ($n \leq 10$), we now generalize to arbitrary n. In the GF quartet Q = (a b c d), we set $d = n$ and find suitable values for c; then b and a are automatically determined as $b = d - c$ and $a = c - b$. A Generalized Fibonacci series $\{G_n\}$ is completely determined by the two starting numbers $(G_0, G_1) = (\alpha, \beta)$. The ratio of consecutive GF numbers attains a limiting value $\varphi = 1.618034$...the Golden ratio which is independent of α and β. α, β are usually chosen as co-prime [gcd $(\alpha,\beta) = 1$]. If α, β are *not* co-prime with gcd (greatest common divisor) g ($\neq 1$), then every G_n is a multiple of g (equation 3). It can be shown that if g is the gcd of any two adjacent numbers in the series, then g divides a. The proof is as follows: If g is the gcd of any pair m, n then g divides $m - n$. Here $g = g(n, c)$ divides $n - c = b$. Since g divides both c and b, g divides $c - b = a$, the smallest number in Q. However, we can cascade down the series to terms $< a$ and finally conclude that g divides β and α. If α or β happens to be 1, then $g = 1$. For Fibonacci and Lucas numbers $\beta = 1$, so $g = 1$.

We have to consider two cases: n odd or n even.

Case I. $n = 2m + 1$ ($= d$).

We choose the quartet Q = (a b c d)

$$Q = (1+2k, \quad m-k, \quad m+k+1, \quad 2m+1) \qquad k = 0, 1, 2, 3.....$$

For instance, for $n = 11$, k determines the quartets

$$k = 0 \qquad Q = (1 \quad 5 \quad 6 \quad 11)$$
$$k = 1 \qquad Q = (3 \quad 4 \quad 7 \quad 11)$$
$$k = 2 \qquad Q = (5 \quad 3 \quad 8 \quad 11)$$

as three possible choices. In all of them adjacent numbers are co-prime. But this may not be always true. For $n = 15$

$$k = 0 \quad Q = (1 \quad 7 \quad 8 \quad 15)$$
$$k = 1 \quad Q = (3 \quad 6 \quad 9 \quad 15)$$
$$k = 2 \quad Q = (5 \quad 5 \quad 10 \quad 15)$$
$$k = 3 \quad Q = (7 \quad 4 \quad 11 \quad 15)$$

For $k = 1$, all numbers in the quartet are divisible by 3 and for $k = 2$ all are divisible by 5. These quartets, however, can also be used for building *kolams* n^2.

Case II: $n = 2m$ $(= d)$

We choose Q = (a b c d) as

$$Q = (2k, \quad m - k, \quad m + k, \quad 2m) \qquad k = 0, 1, 2, 3.....$$

Here a and d are even. But $m + k$ and $m - k$ are either both odd or both even, because their difference is $2k$ an even number. There are two sub-cases to consider: (a) m is even ($= 2t$) (b) m is odd ($= 2t+1$).

Case IIa: $n = 2m = 4t$ $(= d)$

We choose

$$Q = (2k, \quad 2t - k, \quad 2t + k, \quad 4t) \qquad k = 1, 3, 5.....$$

Restricting k to odd values (1, 3, 5,...) ensures that the middle two are odd (unlike a and d). e.g. for $n = 12$

$$k = 1 \quad Q = (2 \quad 5 \quad 7 \quad 12)$$
$$k = 3 \quad Q = (6 \quad 3 \quad 9 \quad 12)$$

For $k = 3$ all numbers in Q are multiples of 3.

Case IIb: $n = 2m = 4t + 2$ $(= d)$

$$Q = (2k, \quad 2t+1- k, \quad 2t+1+k, \quad 4t+2) \qquad k = 0, 2, 4.....$$

Restricting k to even numbers and 0 yields odd numbers for the middle two. e.g. for $n = 10$

$$k = 0 \quad Q = (0 \quad 5 \quad 5 \quad 10)$$
$$k = 2 \quad Q = (4 \quad 3 \quad 7 \quad 10)$$
$$k = 4 \quad Q = (8 \quad 1 \quad 9 \quad 10)$$

In *Figure* 4(e) ($n = 10$), the value of k is 0. Other possible choices are $k = 2, 4$. All the three cases are summarized in Table 1. Note that the gcd of the middle pair of numbers (sides of rectangle) is always a factor of a including 1.

In any GF series with starters (α, β) the ratio of adjacent numbers approaches the Golden ratio $\varphi = 1.618034...$ for large n. However, for a given Q = (a b c d), we can achieve a ratio c/b as close to φ as possible by choosing an optimum value for k. This optimum value k_{opt} is given by

$$k_{opt} = 0.118\, n - 0.5 \qquad (n \text{ odd})$$

$$k_{opt} = 0.118\, n \qquad (n \text{ even})$$

The round off for k_{opt} is as follows. For n odd, it is rounded off to the nearest integer. For $n = 4t$ (a multiple of 4), the round off is to the nearest *odd* integer; for $n = 4t+2$ the round off is to the nearest *even* integer. Table 2 gives the optimum quartet Q (a b c d) and the corresponding k_{opt} for n = 14 to 24.

Before concluding this section, we mention a simple scheme to generate symmetric square *kolams* n^2. It is based on the identity

$$n^2 = (n-2)^2 + 4\,(n-1)$$

Geometrically: square *kolam* n^2 has a central smaller square *kolam* $(n–2)^2$ surrounded on four sides by linear strings of size (n–1) in a cyclic order. Repeating the process, e.g.

$$(n - 2)^2 = (n-4)^2 + 4\,(n-3)$$

etc. one ends up with a 1^2 or 2^2 at the centre. This works for all n, odd or even. Although this appears unrelated to GF numbers, it actually corresponds to

$$Q = (n-2, \quad 1, \quad n-1, \quad n^2)$$

in which b is set equal to 1, to make the rectangle b x c a linear string of length c. For n = 4, the Lucas quartet Q (2 1 3 4) of *Figures* 2a, 2b is an example. Square *kolams* of 5^2 and 6^2 based on this are given in *Figures* 5a, 5b.

4.1 Generalized Fibonacci Rectangular Kolams of Arbitrary Size

Our emphasis has been mostly on GF square *kolams*. We now turn to GF rectangular *kolams* of arbitrary size m x n. Any given rectangle can be broken up into a set of squares and a linear string. This is illustrated by a few examples.

7 x 18 = 7(7+7+4) = 7^2+7^2+4(4+3) = $7^2+7^2+4^2$+3(3+1) = $\mathbf{7^2+7^2+4^2+3^2+(1.3)}$

12 x 19 = 12(12+7) = 12^2+7(7+5) = 12^2+7^2+5(5+2) = $12^2+7^2+5^2$+2(2+2+1)

$= \mathbf{12^2+7^2+5^2+2^2+2^2+(1.2)}$

9 x 13 $= 9(9+4) = 9^2+4(4+4+1) =$ $\mathbf{9^2+4^2+4^2+(1.4)}$.

5 x 8 $= 5(5+3) = 5^2+3(3+2) = 5^2+3^2+2(2+1) =$ $\mathbf{5^2+3^2+2^2+(1.2)}$.

The constituent squares and the linear string have to be positioned in a cyclic pattern as illustrated for 12 x 19 and 7 x 18 in *Figure* 6. The square *kolams* are constructed as described in the previous sections.

5. Enumeration of *Kolams*.

How many distinct *kolams* exist of a given size? This is a problem in combinatorics since the *kolams* are built from smaller sub-units. We start with the smallest square 2^2 single-loop *kolams*. The only symmetric version is the 'cloverleaf' pattern (*Figure* 7a); this is used in building FKs with $a = 2$ in Q (a b c d). The asymmetric versions are useful in building 2 x 3 rectangles (as 2^2+1.2). Four of them are shown in *Figure* 7(b); they are arranged in cyclic order (clockwise rotation by 90^o in succession). Since these are to be used in 2 x 3 rectangles we have to consider all the four as different.

We now proceed to enumerate the different types of 2 x 3 *kolams*. Each 2 x 3 is made up of a 2^2 and a linear string 1 x 2. The constituents need not be single-loop since the single-loop requirement is only for the whole 2 x 3 *kolam*. So, we have more choices for 2^2 as in *Figure* 8(b); the first two of the trio have two loops and the last one has three loops. In addition there are cyclic rotations of the above three as in *Figure* 7(b). For 1 x 2 we have two choices (*Figure* 8a) with one loop and two 'loops' and their cyclic rotations. The 2 x 3 *kolam* pattern will depend not only on the constituents (*Figures* 7,8) but also on the choice of the splicing points to join the 2^2 and 1 x 2 string. Enumeration of all these possibilities can be handled manually. It turns out that there are 30 different 2 x 3 FR *kolams*. They can be classified into six different basic patterns shown in *Figure* 9. Others can be obtained as rotations and reflections of the basic types about suitably chosen axes. The six basic patterns are labeled E R G H U S ; the alphabet patterns reflect the shapes of the *kolams* and have the same symmetry properties as the *kolams*. These

'ERGHUS' *kolams* correspond to the *SaRiGaMaPaDa kolams* [2, 3] discussed in Part I [1].

5.1 Symmetry Properties of Fibonacci Rectangular *Kolams* 2 x 3.

A fundamental concept in science is 'symmetry' which is the subject of Group Theory with far-reaching and wide-ranging applications in all branches of science and technology. Here the operations on the basic pattern (or symmetry operators) are rotations about z-axis (perpendicular to the (X Y) plane of the paper) and reflections (mirror images) in the (X Y) plane. Rotations are labeled R(0^o), R(90^o), R(180^o), R(270^o) – i.e. rotations of 0^o,90^o,180^o,270^o. They are equivalent to I, R(90^o), R(180^o), R(-90^o). I is the 'Identity' operator which leaves the pattern unchanged. Reflection operators are M(X), M(Y), M(45^o), M(-45^o) – i.e. reflections about X-axis, Y-axis, diagonal Y = X and diagonal Y = -X respectively. These 8 operators

I R(90^o) R(180^o) R(-90^o) M(X) M(Y) M(45^o) M(-45^o)

form a 'group' as defined in Group Theory (see Box).

How do these operators alter the shapes of the ERGHUS *kolams* ? Table 3 summarises the results. The basic *kolam* shapes (K) are in the top box. Successive rows show the effect of different operators on K. Instead of the *kolams* of *Figure* 9, we adopt the alphabets which have the same symmetry properties. Some interesting features are as follows;

(a) Of the total 48 (6 x 8) patterns, only 30 are distinct; the remaining 18 are repetitions, shown as dotted patterns.

(b) Only for 'R' and 'G' all the 8 operators yield distinct patterns (columns 2,3). For 'E' 'U' 'S' only four are distinct. For 'H' only two give distinct patterns [I and R(90^o)]. These account for 2 x 8 + 3 x 4 + 1 x 2 = 30 distinct patterns.

(c) From the Group Table (Box), two operators A, B acting in succession on the *kolam* patterns (K) is equivalent to a single operation. e.g. R(90^o) R(90^o) = R(180^o), M(Y) R(90^o) = M(45^o).

The Group theoretic approach could possibly facilitate classification of *kolams* of higher order. This is a topic worthy of further exploration. Another approach to enumeration is Graph Theory. Gift and Rani Siromoney first applied Graph Theory to *kolams* [4]. In the graph, every dot represents a node and a pair of dots linked in the *kolam* is an edge. Using combinatorial techniques the number of possible graphs with a given number of nodes can be counted [5].

5.2 Fibonacci *Kolams* of size 5^2.

In Part I [1] we built 5^2 FS *kolams* using Q = (1 2 3 5); f*igures* 3(b), 3(d) are examples using 'E' as the basic pattern for 2 x 3 rectangles. In *Figure* 3(b) the merging of the four rectangles with the central square (1^2) was by a four-way splice, yielding a single-loop 5^2 FS *kolam*. In *Figure* 3(d), further splices were added at two sets of points (8 in all) to obtain a more complex pattern. Adopting the same strategy for the 30 distinct ERGHUS *kolams* we find the following.

(a) The four-way splice at the centre gives a desired single-loop FS *kolam* for all the 30 patterns.

(b) Additional splices, besides the four-way splice, always yield multiple loops except for pattern 'E'. So, *Figure* 3(d) is unique.

Thus we have 31 (30 + 1) different FS *kolams* 5^2 based on Q (1 2 3 5). Using the Generalized Fibonacci quartet Q (3 1 4 5), a GFS *kolam* 5^2 was made by splicing four linear strings 1 x 3 to a central symmetric 3^2 (*Figure* 5a). This *Figure* is for a particular choice of splicing point; it is clear there are two other choices. Thus we have 3 GF 5^2 *kolams*. For both the quartets put together we have a total of 34 different square *kolams* of size 5^2.

5.3. Fibonacci Square and Rectangular *Kolams* of Higher Order.

The number of possible FKs of size 3 x 5, 8^2 etc rises very rapidly and their enumeration by brute force methods is impractical. We illustrate the complexity of counting for 3 x 5 rectangles. Note that the enumeration problem considered here is restricted to the composition 3 x 5 = 3^2+(2x3). To simplify our attempts

B O X

Basics of Group Theory and Application to *Kolam* Designs.

A Group G is a set of distinct elements G = E_1 E_2 E_3 E_4....) with a 'binary' operation (or 'product') which obeys four rules:

(1) Product of any two elements is also an element of the Group.

If AB = C. then C also belongs to G.

AA = A^2 also belongs to G. This is the 'closure' property.

(2) I, the identity element is a member of G such that

$$I A = A I = I$$

(3) Associative Law: A (BC) = (AB) C

(4) Every element A has an inverse element X in G such that

$$AX = I \text{ or } X = A^{-1}$$

G is an Abelian Group if operators commute: AB = BA.

In the context of symmetry properties of Rectangular *Kolams* we define the Group G of order 8 (8 elements)

$$G = \{I \quad R(90^o) \quad R(180^o) \quad R(-90^o) \quad M(X) \quad M(Y) \quad M(45^o) \quad M(-45^o)\}$$

They are the rotation and reflection operators (section 5). The Group Table represents, as an 8 x 8 matrix, all possible products AB where A and B are any two elements of G (Table 4). Here the 'product' AB has to be interpreted as 'first B, then A' operating on the object. All the four group properties are easily verified in the Table. G is an Abelian Group since AB = BA and the Table is symmetric about the main diagonal. Every column or row is a different permutation of the 8 elements of the Group (closure). For every element A there exists another X such that AX = I and no two elements have the same inverse. Since the order of operations is immaterial the Associative Law is satisfied. Further note the subset of the first four elements of G

$$G_R = \{I \quad R(90^o) \quad R(180^o) \quad R(-90^o)\}$$

also forms a Group obeying all the four properties.

first we consider only single-loop 3^2 and 2 x 3 constituents. For the 2 x 3 rectangles we have 30 choices (single-loop). For 3^2 we consider asymmetric single-loop *kolams* also. These have to be in turn obtained by regarding 3^2 as 2x 3 + 1 x 3.

In [2] 63 *kolams* of 3^2 are displayed. Of these one is a repetition (# 35 and #39 are the same) and 25 have empty unit cells rejected in our analysis. Of the remaining 37, only one is a symmetric single-loop *kolam* (*Figure* 5d, Part I). The remaining 36 asymmetric *kolams* have either one loop (28), two loops (7) or three loops (1). This list may not cover all the possibilities.

In enumerating 3 x 5 rectangles, we have to consider the number of choices for (a) 2 x 3 rectangles (b) 3^2 and (c) splicing points. For (a) we have 30 choices; for (b) up to 28 x 8 (counting reflections and rotations) + 1 (symmetric case), i.e. up to 225 choices. For (c) there are up to 3 choices. Over all there can be up to 30 x 225 x 3 = 20,250 patterns. This is perhaps a gross over-estimate since they include patterns with repetitions, multiple loops and/or empty unit cells. The number is also an under-estimate since only single-loop constituents are considered; multiple loop constituents will yield more possibilities. Finally all this is based only on the composition 3 x 5 = 3^2+ 2 x 3.

Clearly, the enumeration becomes even harder as we go to higher orders. For example to enumerate the number of 8^2 we need to know the number of possible 3 x 5 rectangles which is still uncertain.

6. Fibonacci *Kolams* on Diamond Grids.

Till now we have adopted a square lattice or grid of dots for *Kolam* designs. In the folk art of *Kolam*, diamond-shaped grids are very popular. So we would like to consider extending the Fibonacci *Kolams* to diamond grids.

A diamond grid D(w) is specified by its diameter or width with *w* dots. For odd *w*, the numbers of dots in successive rows from top are 1,3,5,7.......*w*-2*, w, w*-2..........7,5,3,1. In *Figure* 10 a diamond grid is shown for *w* = 13, i.e. D(13). If *w* is even, the pattern of dots in successive rows is 2,4,6.......*w*-2, *w, w, w*-2

......6,4,2. For convenience we consider only odd w. D(w) has a central square grid of size (w+1)/2 surrounded by four identical right-angled triangles on the four sides as shown in *Figure* 10. We set (w+1)/2 = d. Each triangle has a base (the hypotenuse) of (d-2) dots and a height of (d-1)/2 dots. We label this triangle as 'Fibonacci Triangle' FT(d). So far we have dealt with Fibonacci Squares FS(d), Fibonacci Rectangles FR(b,c) related to the quartet Q(a b c d); now we have added a Fibonacci Triangle FT(d) with base (d-2) and height (d-1)/2.

Whatever be the symmetry property of FT(d) the overall symmetry of D(w) is preserved because identical FTs are appended on all the sides. It turns out that FT(d) can be broken up into four parts: (a) a smaller FT(a), (b) an FR(a,b) and (c) two isosceles right triangles RT(b) of height (b-1). By iterating the procedure one can build FT(d) from smaller constituents. Each of them can be individually spliced to the central square grid of side d^2. The same splicing is repeated for all the four FTs, resulting in a single-loop symmetric diamond grid D(w). This scheme is easily adapted for even w too; in this case the FT has two dots in the top row (a trapezium). Further, the procedure works for any w since it is based on Generalized Fibonacci Numbers.

7. Discussion and Summary.

We have presented in this article a general procedure to create square and rectangular *kolam* designs of arbitrary size based on the Generalized Fibonacci numbers. The key feature is the Fibonacci recursion relation (equation 3) that allows building a square (d^2) with sub-units of a smaller square (a^2) and four cyclically placed rectangles (b x c). The construction has built-in four-fold rotational symmetry mandated by our ground rules. Merging the sub-units by splicing requires a judicious choice of splicing points to avoid multiple loops and empty unit cells. First we constructed Fibonacci squares 2^2, 3^2, 5^2, 8^2, 13^2 ,21^2 and Fibonacci rectangles 2 x 3, 3 x 5, 5 x 8 and 8 x 13 in Part I[1] and then extended the list to 4^2, 6^2, 7^2, 9^2 and 10^2 to cover all m^2 ($m \leq 10$). To extend construction to any arbitrary size square, a suitable choice of the quartet of consecutive Fibonacci numbers Q (*a b c d*) has to be made. The problem of enumeration of the number of distinct Fibonacci *kolams* is tractable only for 2 x 3, 3^2 and 5^2 *kolams*; even 3 x 5 *kolams* are too complex to handle manually. The classification of the 30 distinct 2 x 3 *kolams* into 6 classes based on the symmetry properties and the identification of a Group structure of the symmetry operators, may help in the classification of higher order *kolams*. Finally it is shown how the construction can be extrapolated from square to diamond shaped grids.

There may be some interesting connections between *kolams* and knots. Knot theory is a flourishing branch of Algebraic topology with many applications in science and technology. A knot is a tangled loop(s) of curve(s) in 3-dimensional space and is characterised by *n* the number of crossings (over and under). The simplest knot is the 'unknot' which is a single loop and many such loops will form a link. The single-loop *kolam* is actually the trivial unknot and multiple loop *kolam* is equivalent to multiple unknots not linked with each other. The most important and unsolved problem in Knot theory is to find an algorithm to determine if two given knots of *n* crossings are topologically equivalent or not. The corresponding problem in *kolams* is to decide if two given *kolams* of the same size have the same number of loops, which may be a trivial problem. However, it is possible that Knot theory and Group theory, besides Graph Theory, may be of help in classifying and enumerating *kolams* of various sizes.

Acknowledgments

I am grateful to T.V. Suresh for invaluable help with computer graphics software for drawing some of the *kolams*. The two supplements to Tamil magazines [2,3] were the starting points and inspiration for this work. On a personal note: my daily morning walks in the by-lanes of South Chennai made me aware of the beauty, artistry and ingenuity in the *kolams* drawn in front of the house-holds and every day brings surprises. As a lad of 8 years, I was taught to draw *kolams* by my mother and I can still recall the thrill of learning to draw *Brahmamudi.* This article is dedicated to her memory.

Suggested Reading:

[1] S. Naranan. *Kolam* designs based on Fibonacci numbers. Part I: Square and Rectangular Designs. *Resonance* (2007)

[2] Salem Meera. Supplement to Tamil magazine '*Kumudam Snehidi',* December 2005.

[3] Sumati Bharati. *Sarigamapadani Kolangal* Supplement to Tamil magazine '*Snehidi',* February 2007.

[4] Gift Siromoney and Rani Siromoney. *Mathematics for the Social Sciences.* Chapter 6. National Book Trust of India (New Delhi), Reprint (1979).

[5] R.Kit. Kittappa (private communication).

Figure Captions.

Figure 1. Construction of a square Fibonacci *kolam*. (Figure 2 of Part I).

Figure 2. Lucas *kolams*: 4^2.

Figure 3. Lucas *kolams*: 7^2 and 9^2.

Figure 4. Generalized Fibonacci *kolams* (GFKs): (a), (b) 6^2 (c), (d) 9^2.

Figure 5. Generalized Fibonacci *kolams*: (a) 5^2 (b) 6^2.

Figure 6. Composition of GFK rectangles: (a) 12 x 19 (b) 7 x 18.

Figure 7. Fibonacci *kolams*: 2^2 (single loop)

Figure 8. Fibonacci *kolams*: 2^2 (multiple loop).

Figure 9. Fibonacci Rectangular *kolams*: 2 x 3

Figure 10. Structure of Diamond Grid *kolam* D(13).

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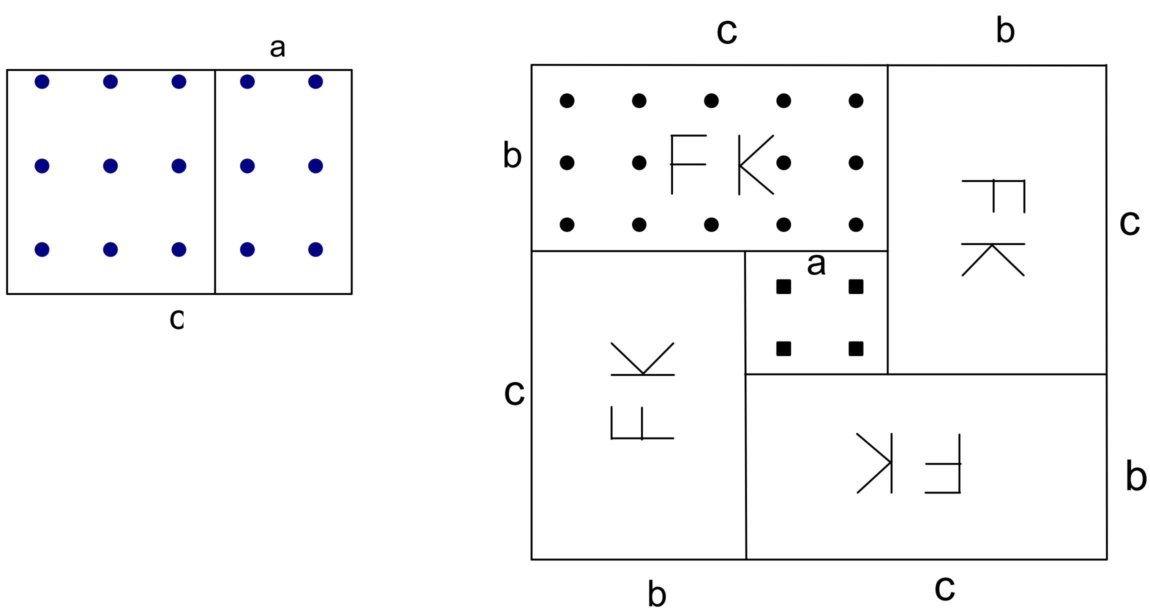


FIGURE 1

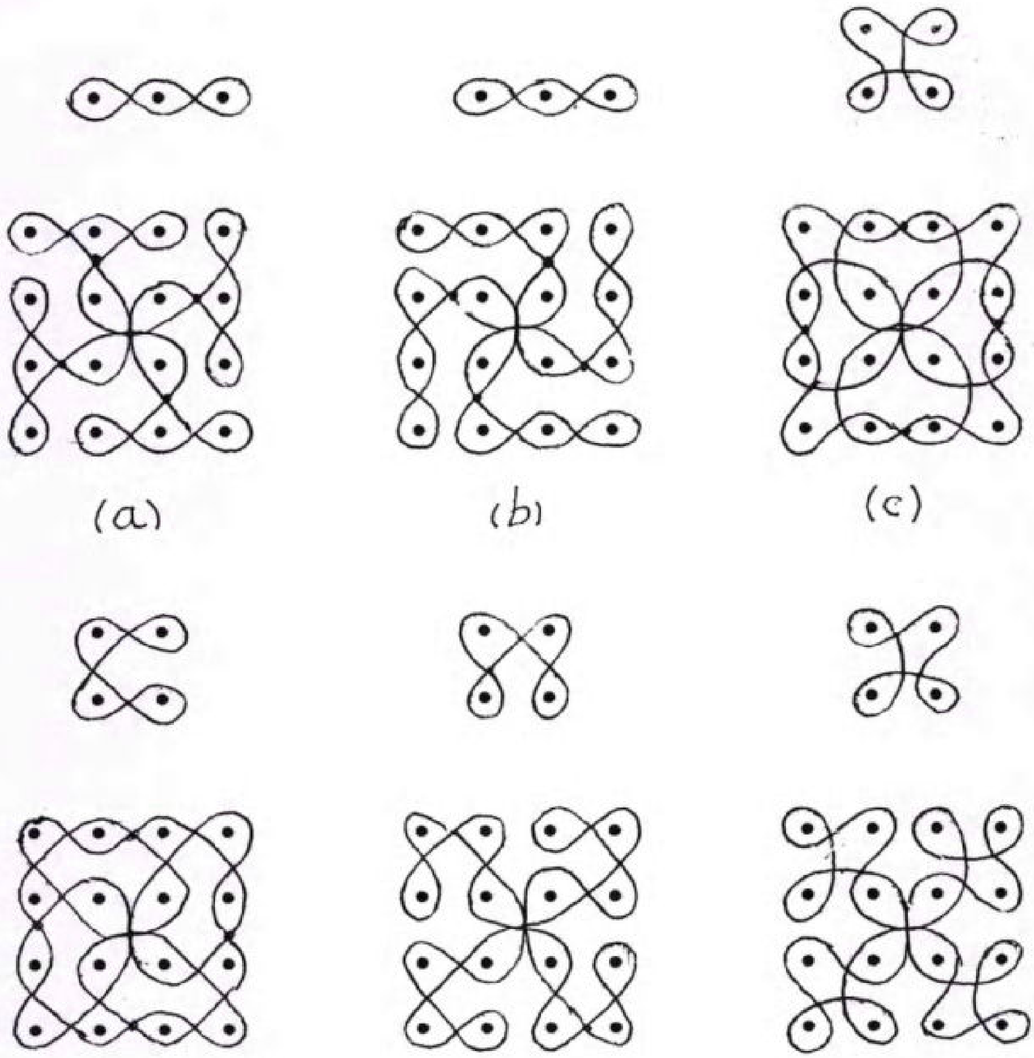


Figure 2.

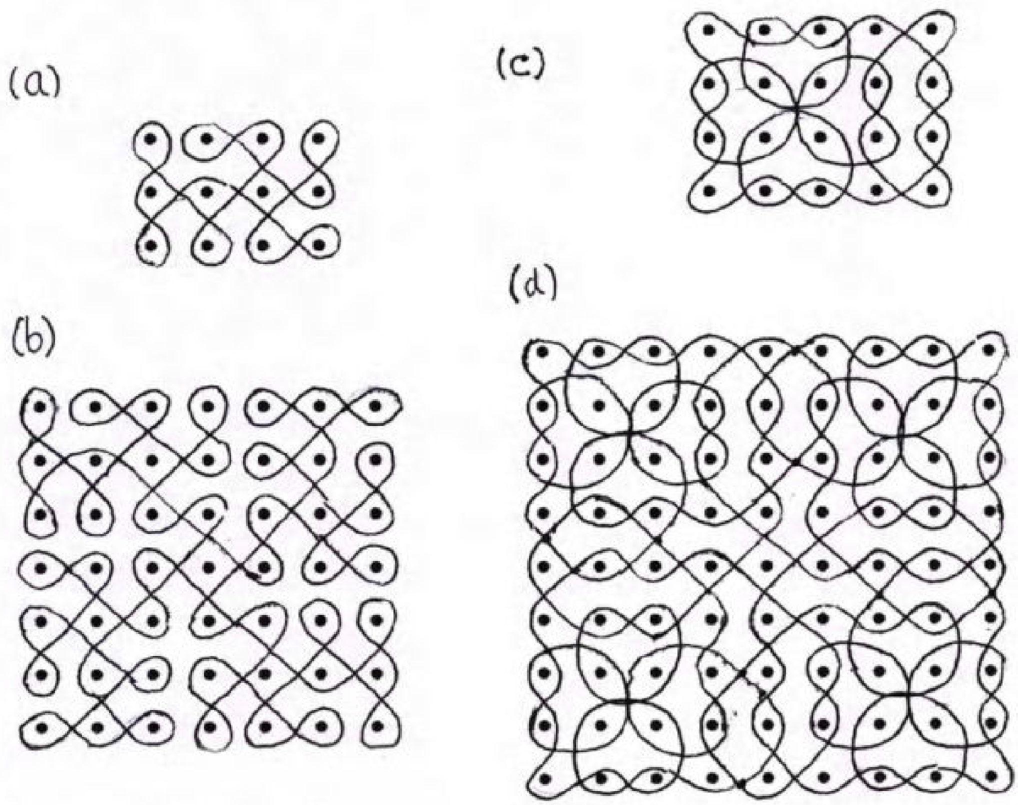


Figure 3

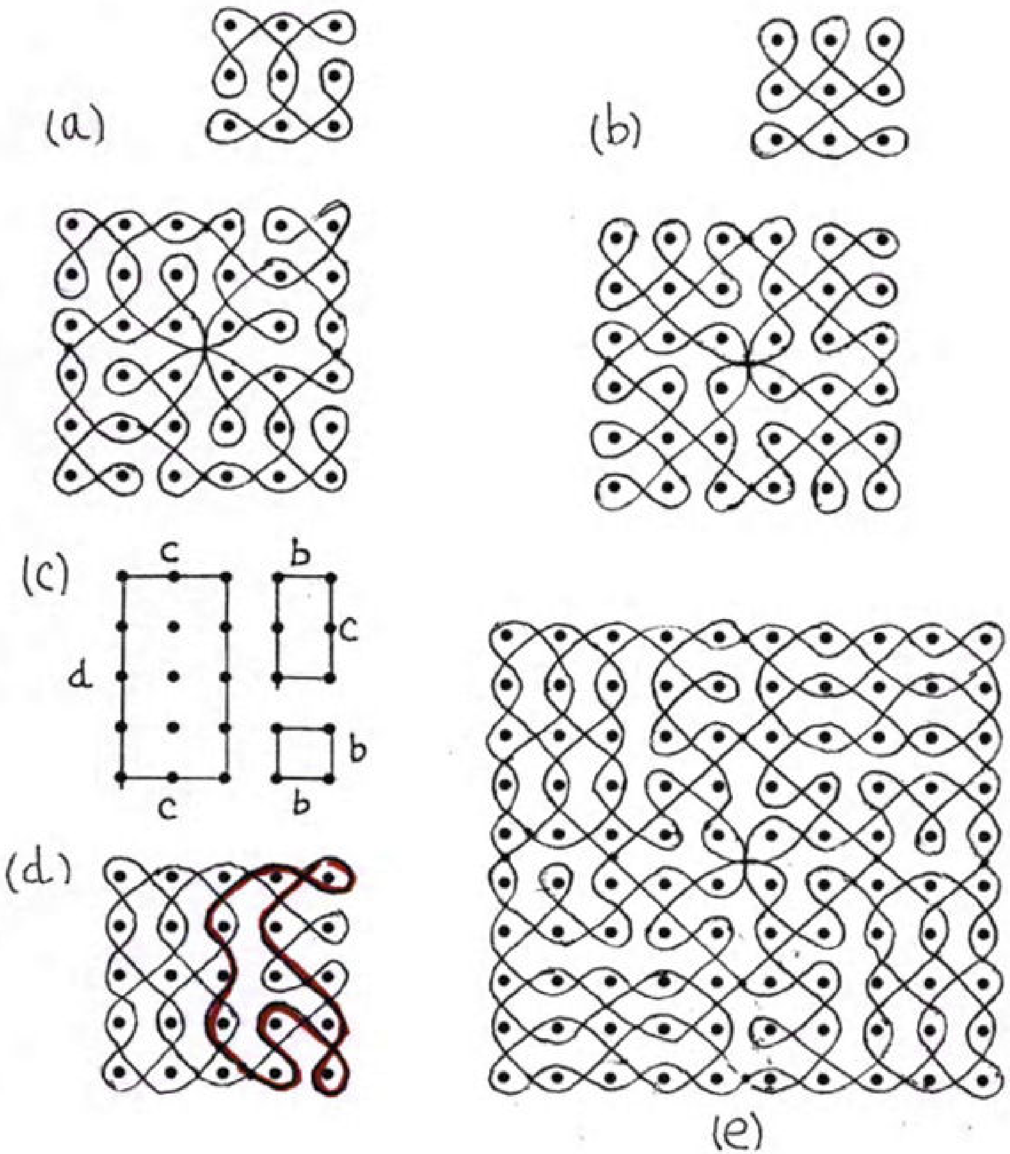
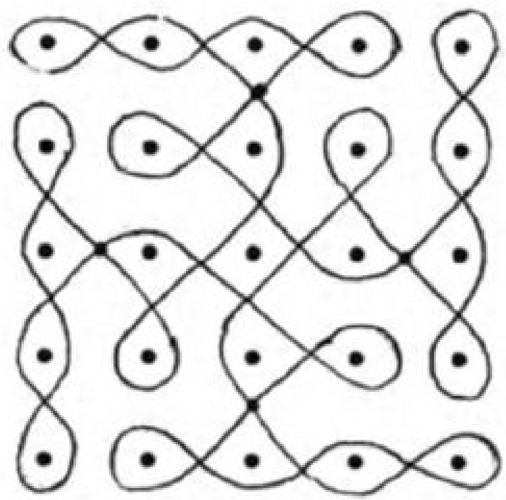
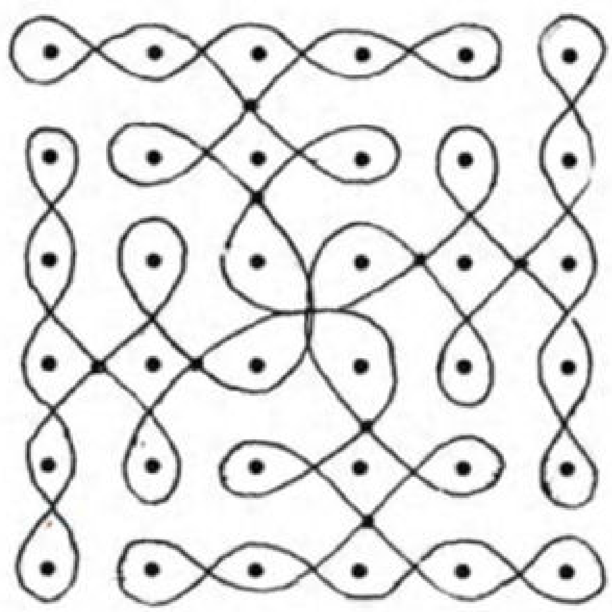


Figure 4



(a)



(b)

Figure 5

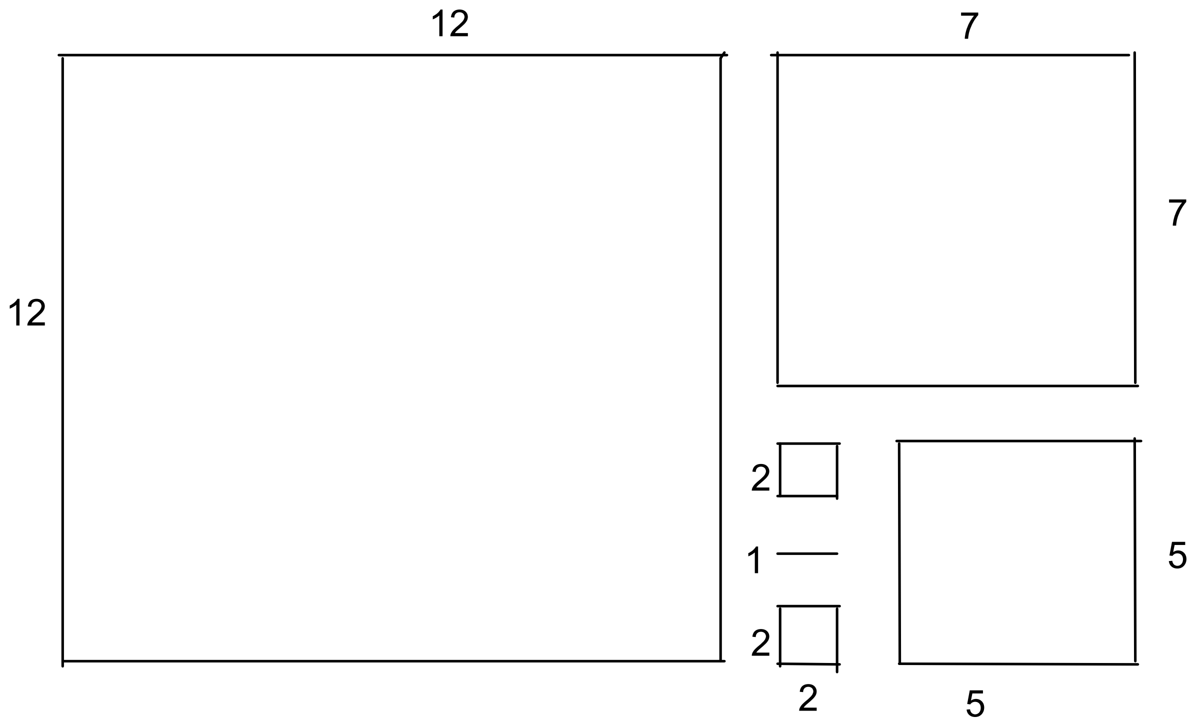


Fig 6a. COMPOSITION OF 12 X 19 RECTANGLE

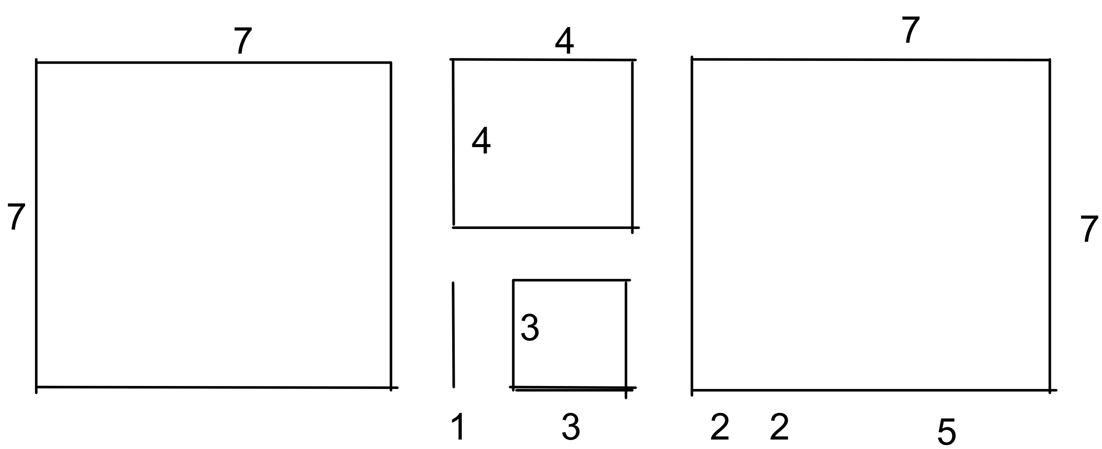


Fig 6b. COMPOSITION OF 7 X 18 RECTANGLE

FIGURE 6

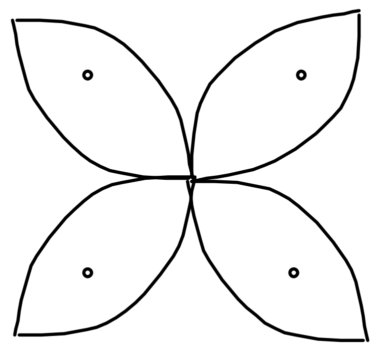


FIGURE 7(a)

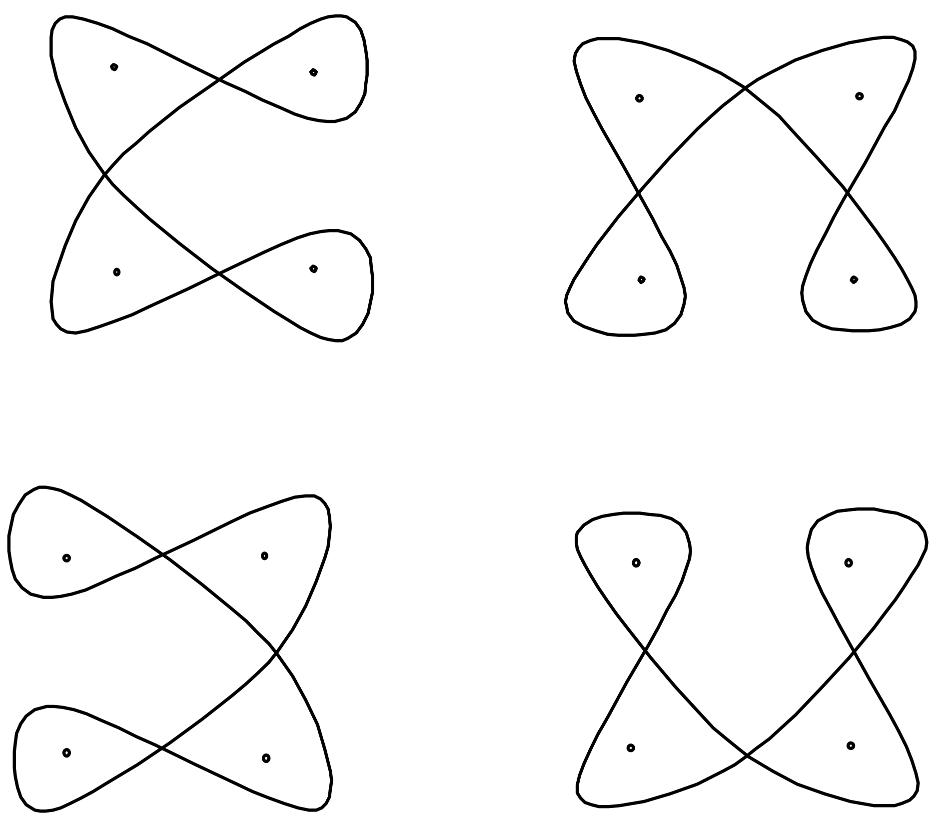


FIGURE 7(b)

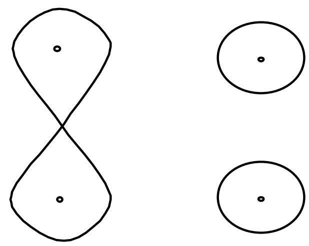


FIGURE 8(a)

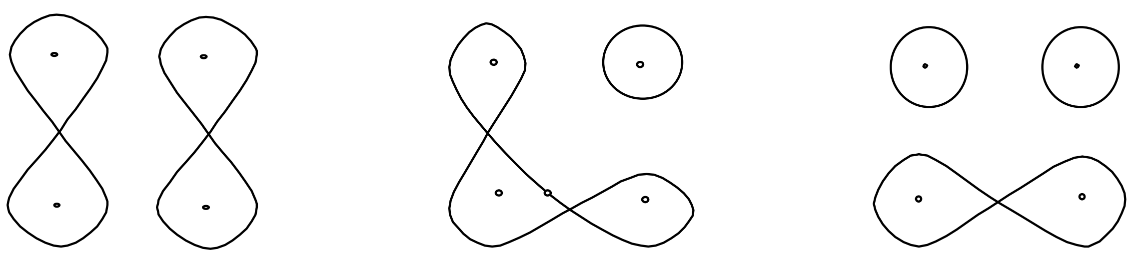


FIGURE 8(b)

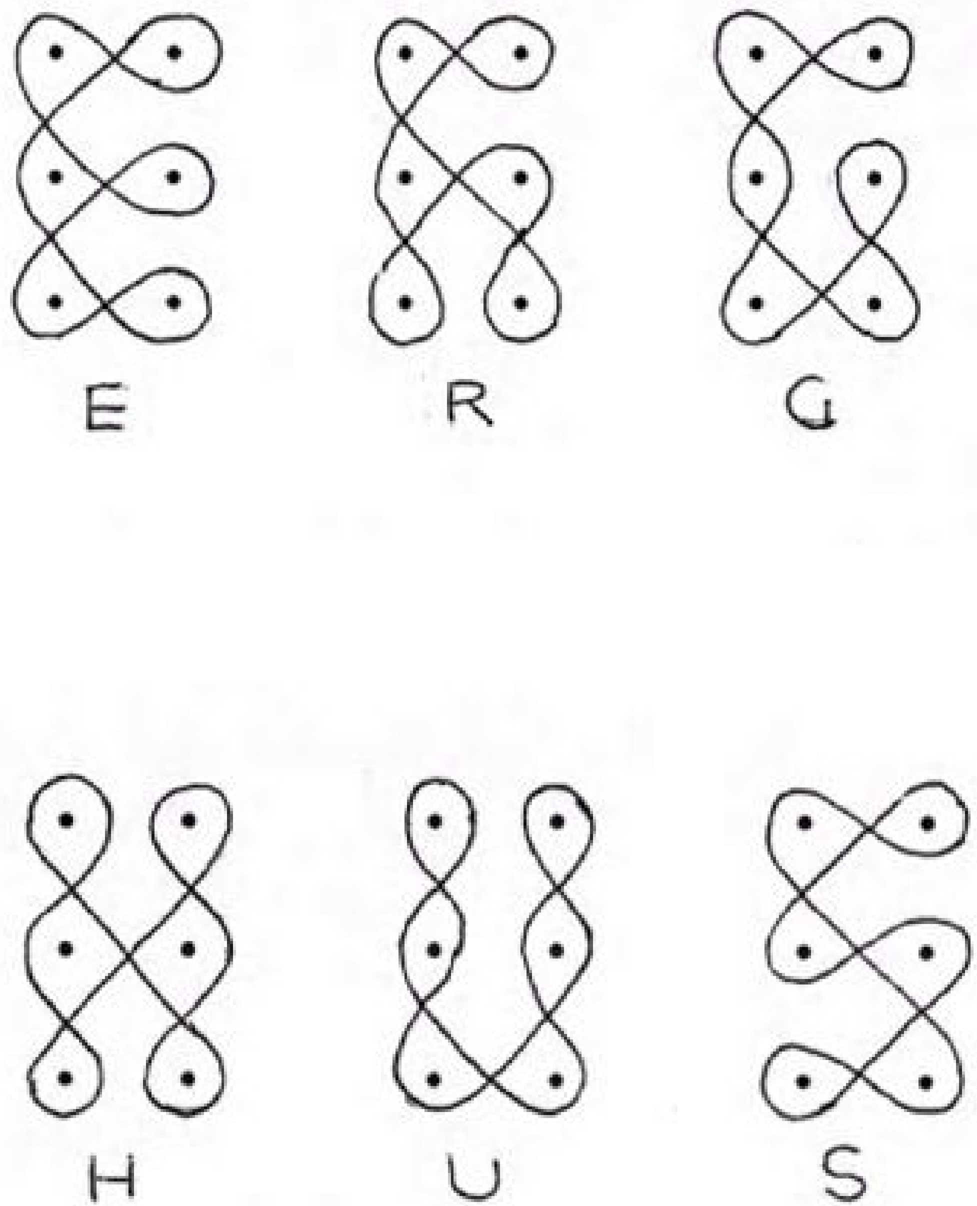


Figure 9

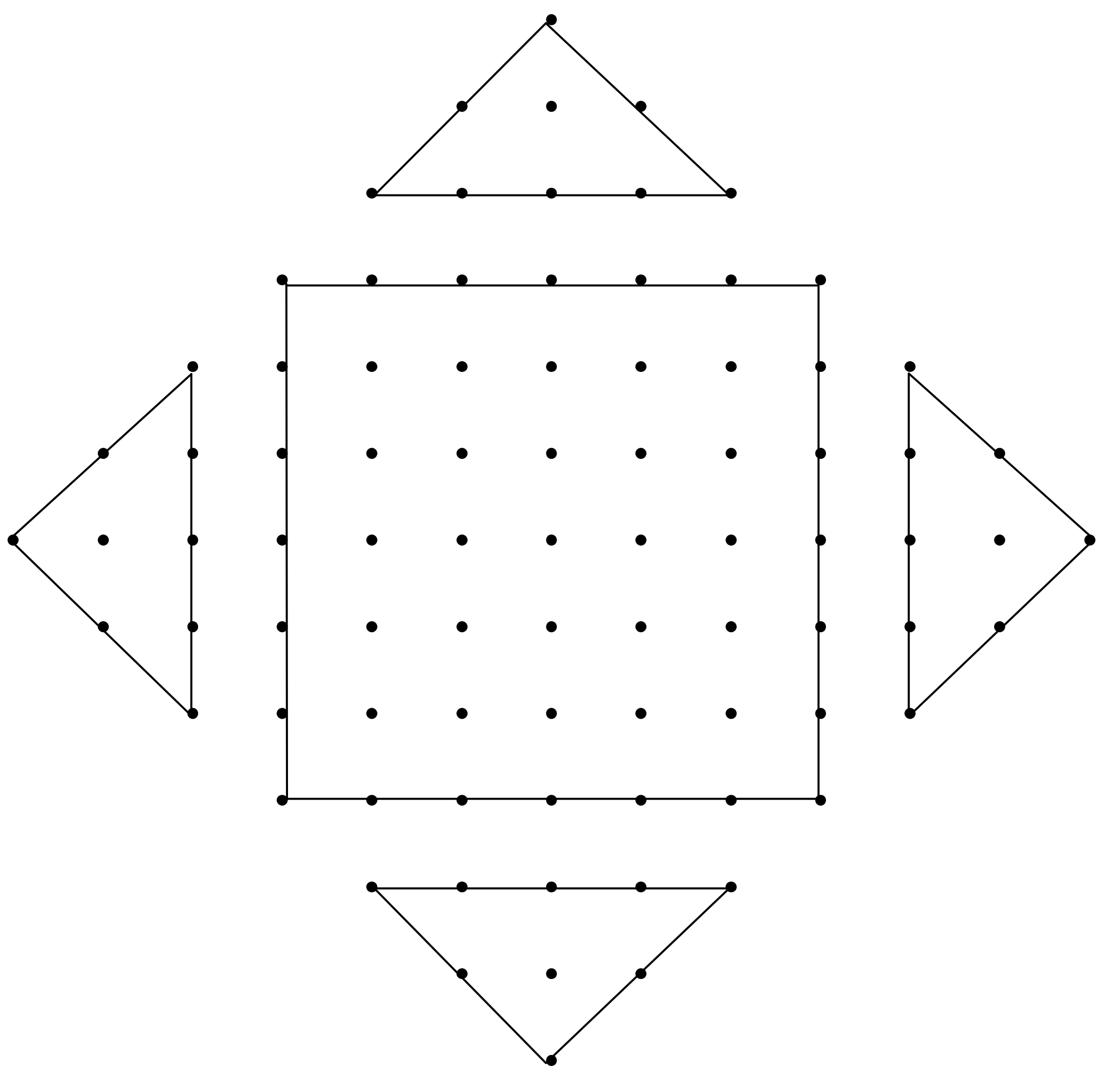


FIGURE 10: STRUCTURE OF THE DIAMOND GRID

Table 1: Summary of Q = (a b c d) for Generalized Fibonacci Square Kolams of size *n*

n	a	b	c	d	k
$2m+1$ **(m = 1,2..)**	$1+2k$	$m-k$	$m+k+1$	$2m+1$	0,1,2,3.
$4t$ **(t = 1,2…)**	$2k$	$2t-k$	$2t+k$	$4t$	1,3,5…
$4t+2$ **(t = 1,2…)**	$2k$	$2t+1-k$	$2t+1+k$	$4t+2$	0,2,4,6..

Table 2: Choices of Q = (*a b c d*) and k_{opt} for some n

n	k_{opt}	a	b	c	d = n
14	2	4	5	9	14
15	1	3	6	9	15
16	1	2	7	9	16
17	2	5	6	11	17
18	2	4	7	11	18
19	2	5	7	12	19
20	3	6	7	13	20
21	2	5	8	13	21
22	2	4	9	13	22
23	2	5	9	14	23
24	3	6	9	15	24

TABLE 3. SYMMETRY PROPERTIES OF 2 X 3 RECTANGULAR KOLAMS

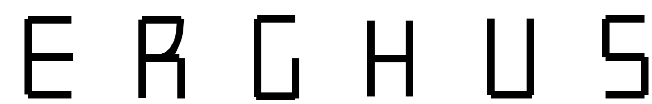
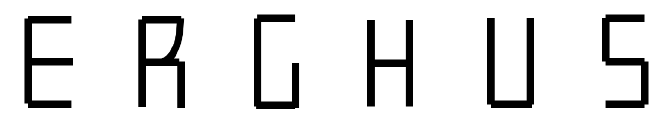
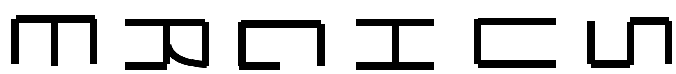
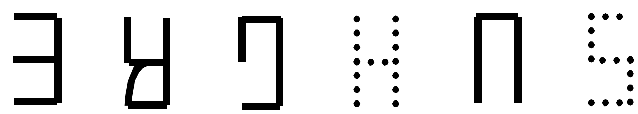
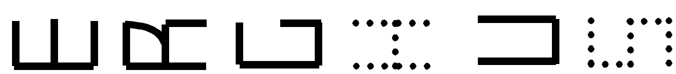
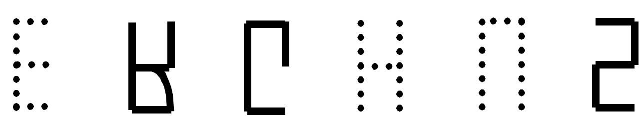
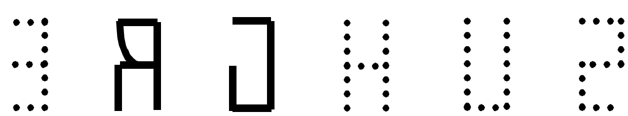
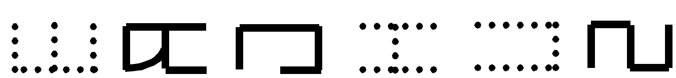
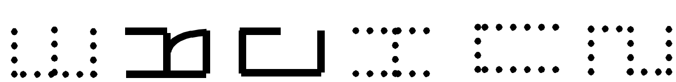
SYMMETRY OPERATION	KOLAM K	
I	(X Y)	
R(90)	(Y -X)	
R(180) =Mo	(-X -Y)	
R(-90)	(-Y X)	
Mx	(X -Y)	
My	(-X Y)	
M(45) =MyR(90)	(Y X)	
M(-45) =MxR(90)	(-Y -X)	

TABLE 4. GROUP TABLE:
SYMMETRY OPERATORS OF RECTANGLES

		I	**R(90)**	**R(180)**	**R(-90)**	**M(x)**	**M(y)**	**M(45)**	**M(-45)**
(X Y)	**I**	I	R(90)	R(180)	R(-90)	M(x)	M(y)	M(45)	M(-45)
(Y -X)	**R(90)**	R(90)	R(180)	R(-90)	I	M(-45)	M(45)	M(y)	M(x)
(-X -Y)	**R(180)**	R(180)	R(-90)	I	R(90)	M(y)	M(x)	M(-45)	M(45)
(-Y -X)	**R(-90)**	R(-90)	I	R(90)	R(180)	M(45)	M(-45)	M(x)	M(y)
(X -Y)	**M(x)**	M(x)	M(-45)	M(y)	M(45)	I	R(180)	R(-90)	R(90)
(-X Y)	**M(y)**	M(y)	M(45)	M(x)	M(-45)	R(180)	I	R(90)	R(-90)
(Y X)	**M(45)**	M(45)	M(y)	M(-45)	M(x)	R(-90)	R(90)	I	R(180)
(-Y -X)	**M(-45)**	M(-45)	M(x)	M(45)	M(y)	R(90)	R(-90)	R(180)	I