KOLAM DESIGNS BASED ON FIBONACCI NUMBERS

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Part III. Rectangular *Kolams* with two-fold Rotational Symmetry

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A B S T R A C T

In Parts I and II a general scheme was given to create square *kolam*s based on 'Fibonacci Recurrence'. Based on the canonical Fibonacci Series

0 1 1 2 3 5 8 13 21 34.....

Square *kolam*s 3^2 5^2 8^2 13^2 21^2 were presented in Part I. Square *kolam*s of any desired size can be generated based on the Generalized Fibonacci Series (Part II). Rectangular *kolam*s with sides as consecutive Fibonacci numbers – the Golden Rectangles – were also drawn, but they lack any symmetry property unlike the Square *kolam*s which have four-fold rotational symmetry. In this paper (Part III), Rectangular *kolam*s with two-fold rotational symmetry based on Fibonacci Recurrence are presented.

1. Introduction.

*Kolam*s are geometrical designs, line drawings of curves around a basic template of grid of points. They are a part of the folk art in South India, adorning homes and temples. Of all the variants of the art, the *Pulli Kolam* (*Pulli* = dot) is the most popular. The square *kolam*s based on a square grid of points are generally required to have four-fold rotational symmetry. This means the *kolam* viewed from all the four sides North, East, South and West, appears the same. The curving lines of the *kolam* may be a single loop or multiple loops. But the single loop *kolam*s are special in their aesthetic appeal and are harder to achieve.

In two earlier papers '*Kolam Designs based on Fibonacci Numbers, Parts I and II'* [1][2], I had presented a general scheme to create square *kolam*s of any arbitrary size based on 'Fibonacci Recurrence'. In Fibonacci Recurrence, a series of integers is generated in which every number is a sum of the two immediately preceding numbers. (Clearly, the first two numbers are free to choose). The rationale for the scheme is that the four-fold symmetry is automatically satisfied, as it is built into the Fibonacci Recurrence. In Part I, the *kolam* designs were based on the well known 'Fibonacci Series' in which the first two (starting) numbers are 0, 1:

0,1 1 2 3 5 8 13 21 34 55 89 144 ....

Square *kolam*s 3^2 5^2 8^2 13^2 and 21^2 were presented. In Part II, square *kolam*s of any desired size are created based on 'Generalized Fibonacci Series', by appropriate choice of the starting pair of numbers.

Let Q (a b c d) be a quartet of four consecutive numbers in a Generalized Fibonacci Series. The Fibonacci Recurrence implies

$$c = a + b \qquad d = c + b = a + 2\,b \tag{1}$$

An equation relating a b c d is

$$d^2 = a^2 + 4\,b\,c. \qquad (2)$$

A geometrical interpretation of the above is the following. A square of side *d* has at its centre a smaller square of side *a* and four rectangles of sides *b, c* placed in a cyclical pattern to fill the space within the two squares (Figure 1). In the figure the quartet chosen is Q (2 3 5 8). It is important to note that no symmetry property is required of the rectangles (this is illustrated by deliberate choice of an 'asymmetric' rectangle with 'FK'). But the central square a^2 is required to have four-fold rotational symmetry. Figure 1 looks the same viewed from all the four sides (four-fold rotational symmetry around a vertical axis perpendicular to the plane of the *kolam*). Equation (2) follows easily from equation (1).

$$d - a = 2\,b \qquad d + a = 2\,a + 2\,b = 2\,c$$

Multiplying

$$(d - a)\,(d + a) = (2\,b)\,(2\,c)$$
$$d^2 - a^2 = 4\,b\,c$$

I had also presented rectangular *kolam*s with sides equal to a pair of consecutive Fibonacci numbers *b,c.* These are called Golden Rectangles since in any Generalized Fibonacci Series the ratio of consecutive numbers tends to the Golden Ratio $\varphi = (1 + \sqrt{5})/2 = 1.61803....$ a limiting value. These rectangles consist of a square and a rectangle spliced together (Figure 1). A rectangle can have only two-fold rotational symmetry (unlike squares with four-fold rotational symmetry). But the rectangular *kolam*s described do not have any symmetry. In this paper I present a scheme to create rectangular *kolam*s with two-fold symmetry based on Fibonacci Recurrence. This requires two Fibonacci quartets. Paradoxically the Fibonacci Rectangles are harder to create than Fibonacci Squares.

2. Rectangular *Kolam*s with two-fold rotational symmetry.

In analogy to equation (1), we have two Fibonacci Quartets.

$$Q_1\ (a_1\ \ b_1\ \ c_1\ \ d_1)$$

$$Q_2\ (a_2\ \ b_2\ \ c_2\ \ d_2)$$

$$c_1 = a_1 + b_1 \qquad d_1 = c_1 + b_1 = a_1 + 2\ b_1$$

$$c_2 = a_2 + b_2 \qquad d_2 = c_2 + b_2 = a_2 + 2\ b_2 \tag{3}$$

An equation relating all the numbers is

$$d_1\, d_2 \ = \ a_1\, a_2 \ + 2\ b_1\, c_2 \ + 2\ c_1\, b_2 \tag{4}$$

A geometrical interpretation of the above is the following. A rectangle (of area) $d_1\, d_2$ (height d_1, width d_2) has a smaller rectangle $a_1\, a_2$ at the centre; the space between them is filled with two pairs of rectangles, $b_1\, c_2$ and $c_1\, b_2$ (Figure 2). In the figure the two rectangles $b_1\, c_2$ (indicated by hashed lines) are symmetrically placed so that one is rotated 180$^{\mathrm{o}}$ with respect to the other. The same applies for the two rectangles $c_1\, b_2$. This ensures that the figure appears the same viewed from North and South *or* from East and West (two-fold rotational symmetry) provided the central rectangle has a similar property. In the figure the numerical values of the quartets are

$$Q_1\ (3\ \ 2\ \ 5\ \ 7)$$

$$Q_2\ (5\ \ 3\ \ 8\ \ 11)$$

[check: 7 x 11 = 3 x 5 + 2 (2 x 8) + 2 (5 x 3)]. Equation (4) follows from equation (3).

$$\begin{aligned} d_1\, d_2 \ &= \ (c_1 + b_1)\ (c_2 + b_2) \\ &= (\ a_1 + 2\ b_1)\ (a_2 + 2\ b_2) \\ &= \ a_1\, a_2 + 2\ b_1\, a_2 + 2\ b_2\, a_1 + 4\ b_1\, b_2 \\ &= \ a_1\, a_2 + 2\ b_1\, (a_2 + b_2) + 2\ b_2\, (\ a_1 + b_1) \end{aligned}$$

$$d_1\, d_2 \ = \ a_1\, a_2 + 2\ b_1\, c_2 + 2\ c_1\, b_2 \tag{4}$$

The partition of the rectangle d_1d_2 into five rectangles in the figure is indicated by arrows connecting the two quartets of numbers.

Two rectangular *kolam*s 4 x 7 and 3 x 8 are illustrated in Figure 3. For 4 x 7 the quartets are

$$Q_1\ (2\ \ 1\ \ 3\ \ 4)$$
$$Q_2\ (3\ \ 2\ \ 5\ \ 7)$$

At the centre is the 2 x 3 *kolam*. The rectangle 1 x 5 - here a linear string - and the rectangle 3 x 2 appear in pairs surrounding the central rectangle, to make up the larger rectangle 4 x 7. The five rectangles are spliced together at five pairs of points AA BB CC DD EE. The result is shown in the second figure from the top. The four small arrows on the sides when extended into the *kolam* as dotted lines, delineate the five constituent rectangles; the splicing points are marked by small circles and they occur along the lines. Two essential requirements for the construction are (a) the splices are symmetrically placed pairs (AA BB .....) (b) the central rectangle has two-fold rotational symmetry. They ensure the over-all two-fold rotational symmetry of the 4 x 7 *kolam*. Note that the *kolam* is single-loop. The 3 x 8 *kolam* has at the centre a linear string 1 x 4. The four pairs of splices (AA BB CC DD) yield a single loop.

Three more *kolam*s 5 x 6, 5 x 9 and 6 x 9 are presented in Figures 4 and 5. In 5 x 6 only the finished *kolam* is shown; the four pairs of splices are indicated. The constituent rectangles can be inferred from the small arrows. The rectangle 5 x 9 is noteworthy since one of the constituents is a 3^2. In the 6 x 9, as in the 4 x 7 (Figure 3), the 2 x 3 rectangle at the centre has two-fold rotational symmetry.

3. Choice of the Fibonacci Quartets.

Rectangular *kolam*s with two-fold rotational symmetry can be drawn with any desired dimensions. This means that d_1 and d_2 in the quartets Q_1

and Q_2 can be any desired integers. In a Fibonacci quartet Q ($a\ b\ c\ d$), if any two of the four integers $a\ b\ c\ d$ are known the other two can be determined. Equation (1) gives c, d in terms of a and b.

$$c = a + b \qquad d = c + b = a + 2\,b \tag{1}$$

Alternately, given d and a

$$b = (d - a)/2 \qquad c = (d + a)/2$$

Since b and c are integers, a and d should be of same parity. They are *both odd* or *both even*. Then b and c are fixed as $(d - a)/2$ and $(d + a)/2$. If the sides d_1 and d_2 are specified, the sides of the central rectangle a_1 and a_2 have to be chosen such that a_1 and d_1 are of same parity and a_2 and d_2 are also of same parity. This can be verified in the *kolam*s presented in Figures. 3, 4 and 5.

To illustrate the above, we choose 6 x 7 for the rectangular *kolam* with central rectangle 4 x 1 in Figure 6. The arrows help identify the five constituents and the five pairs of splicing points. This may be called "Kaprekar *kolam*" since the numbers are the digits of the Kaprekar constant 6 1 7 4, a unique four-digit number with interesting properties [3][4].

It is noteworthy that a_1 or a_2 can be 0. In such case there is no central rectangle. In Figure 7 are given the smallest non-trivial rectangular *kolam*s 2 x 3 and 3 x 4. In 2 x 3 *kolam* $a_1 = 0$. Five possible splicing points are shown. Only three splices result in single loop and there are two such *kolam*s. In my earlier work [2], 2 x 3 *kolam*s were built up by splicing a 2^2 and 1 x 2. It was shown that there are only six basic single-loop 2 x 3 *kolam*s labelled E R G H U S . It turns out that only two of them H and S have two-fold rotation symmetry and both appear in Figure 7.

In the 3 x 4 *kolam* $a_2 = 0$. Five different *kolam*s are shown all with five splices. (The number of splices is odd because there is no central rectangle). The first four have the same 2^2 in different orientations; the last has the clover-leaf pattern introduced in [1]. Small *kolam*s like the 2 x 3 and the 3 x 4 are very useful as constituents of larger *kolam*s. This modular construction is the key strategy for building larger *kolam*s. For example 2 x 3 *kolam*s are used to build a 4 x 6 *kolam* (Figure 8). Here the central rectangle is a 2^2 clover-leaf and the 2 x 3 rectangles are the ERGHUS *kolam*s with orientation indicated at the South-East corner. The number of splices is 2 x 5 = 10 or 2 x 4 = 8, as indicated.

4. Choice of Splicing Points

Several *kolam*s have been described so far in which five rectangles or five loops are merged together to create a single large rectangle of one loop. The merger occurs at 'splicing points' along the edges of the rectangles. The choice of these points is crucial since they determine the number of loops. In the *kolam*s described the splicing points are chosen specially to yield a single loop. There is one other restriction imposed (rather self-imposed) on the splices, mainly to restrict the number of possible permutations of splices which can be very large especially as the size of *kolam*s becomes bigger.

To illustrate the close connection between the splices and the loops, in Figure 9 we trace the 'evolution of loops' beginning with a 3 x 5 five-loop *kolam* (a) and ending as a single-loop *kolam* (g) after four pairs or 8 splices (A_1A_2 B_1B_2 C_1C_2 D_1D_2). The splicing points are shown in (b). The four rectangles are merged with the central rectangle at A_1A_2 B_1B_2 yielding a single loop (c). The successive *kolam*s show the changes as splices are added at C_1C_2 D_1D_2. A splice at C_1 splits the *kolam* into two loops (d), the splice at C_2 further splits it into three loops (e). Splice at D_1 merges two of the three loops resulting in two loops (f). Finally the splice at D_2 further reduces to one loop. The loops are colour-coded. The above evolution of loops is represented by

$$5 \rightarrow A_1A_2B_1B_2 \rightarrow 1 \rightarrow C_1 \rightarrow 2 \rightarrow C_2 \rightarrow 3 \rightarrow D_1 \rightarrow 2 \rightarrow D_2 \rightarrow 1$$

This can be understood in terms of two basic splicing rules:

(a) a splice *between* two loops gives one loop and

(b) a splice *within* a single loop splits it into two loops.

In my experience, the second rule works most of the time, although a splice within a single loop can in principle leave the loop single.

There are more splicing points available along the edges of the rectangles but they are not allowed by the restriction referred to. For example splices at S_1S_2 [shown in (b)] are disallowed for the following reason: a splice at S_1 creates a four-sided closed 'island' without a central dot, between A_1 and S_1 (same is true for the splice at S_2). It is found that when adjacent dots are used as splices, the islands – the empty spaces between them - almost always are four-sided without a dot.

Even with the restriction there is a wide choice of splices. Once a set of splices is chosen all the splices can be made in one single attempt. One will end up with one or more loops. For large *kolam*s it is not feasible to experiment with different sets of splices; a practical strategy is to choose a set of nearly maximal or maximal number of splices. If the result is a single loop, the work is done. If there are multiple loops, one has to remove or unsplice some points. The rules of unsplicing are

(a) unsplicing at the *intersection* of two loops leads to one loop and

(b) unsplicing within a *single* loop will split it into two loops.

If the intent is to reduce the number of loops, rule (a) is invoked; then it would seem that rule (b) is not useful. However since unsplicing is done in symmetrical pairs (like $A_1A_2B_1B_2$ ......) one may be forced to unsplice within a loop and end up again with two loops. Therefore in practice unsplicing choices are not always easy. There are instances where it is hard to achieve a single loop. Armed with these guiding principles in the choice of Fibonacci Quartets and splicing points, we now try some larger *kolam*s.

5. Pi-*kolam* and e-*kolam*

Pi-*kolam* is a rectangle of sides 7 x 22, with ratio 22/7 = 3.1428..... a very good approximation to π (*Pi*) = 3.14159.., the ratio of the circumference and diameter of a circle. *e-kolam* has sides 7 x 19 with ratio 19/7 = 2.7142 ... a close approximation the Euler's constant e (2.71828.......) best known as the natural base for logarithms. Both π and e and the Golden Ratio φ are regarded as among the most important of mathematical constants.

The π-*kolam* can be coded by the quartets

$$Q_1\ (1\ \ 3\ \ 4\ \ 7)$$
$$Q_2\ (6\ \ 8\ \ 14\ \ 22)$$

At the centre is the linear string 1 x 6. The surrounding rectangles are 3 x 14 and 4 x 8, each occurring in pairs. In Figure 10A are shown the 4 x 8 and 3 x 14 *kolam*s which will serve as modules for the 7 x 22 *kolam* in Figure 10B. The 4 x 8 is composed of a 2^2 (clover-leaf), 3^2 and 1 x 5 modules. The 3 x 14 is composed of a linear string 1 x 4 and 1 x 9 and 2 x 5 modules. All the modules can be inferred with the help of arrows along the four sides. In Figure 10B, the 7 x 22 is assembled from the 3 x 14 and 4 x 8 modules with 2 x 10 = 20 splices. The splices occur along the lines indicated by the arrows. Two more splices AA, which are allowed, split the single loop *kolam* into three loops. Figure 10C is a skeleton diagram of 7 x 22 showing the details of construction of the modules described in Figures 10A, 10B.

The *e-kolam* is based on quartets

$$Q_1\ (1\ \ 3\ \ 4\ \ 7)$$
$$Q_2\ (5\ \ 7\ \ 12\ \ 19)$$

At the centre is the linear string 1 x 5. The four rectangles are 3 x 12 and 4 x 7 which are assembled as shown in Figure 11A. They serve as modules for the 7 x 19 in Figure 11B. (Note that the 4 x 7 *kolam*s used here differ from the 4 x 7 of Figure 3 in the choice of splices). There are 2 x 8 = 16 splices. Two additional splices at AA split the single loop into three loops. *Pi*- and *e*-*kolam*s devoid of any symmetry were presented as 'special *kolam*s and non-standard *kolam*s' in [5].

6. Window-frame *Kolam*

In Figure 12B is drawn a 'regular' 9 x 13 *kolam* based on

$$Q_1\ (5\ \ 2\ \ 7\ \ 9)$$

$$Q_2\ (7\ \ 3\ \ 10\ \ 13)$$

The constituents 5 x 7, 2 x 10 and 7 x 3 are shown in Figure 12A. The number of splices is 2 x 10 = 20.

A simple variant of the 9 x 13 is a 'Window-frame' *kolam* shown in Figure 13A. The central rectangle 5 x 7 is omitted. The number of splices is reduced from 20 to 8. But the most notable feature is that it has two loops (shown in blue and red). In Figure 13B a different configuration of the four rectangles is shown, again with 8 splices. The result is again two loops (red and green). It can be shown that whatever be the quartets and splices, a window-frame *kolam* can never be a single loop. Is this related to the fact that there is no central rectangle? The answer is 'yes'. Since the starting configuration has four loops (corresponding to four rectangles), an even number, parity conservation dictates that the minimum number of loops is two. For details see Section 8.

7. Square *Kolam*s

As mentioned in the Introduction, Square *kolam*s with four-fold rotational symmetry are based on Fibonacci Recurrence and many examples are to be found in [1][2]. However Square *kolam*s (d x d) with two-fold rotational symmetry are possible with two quartets

$$Q_1\,(a_1\;\; b_1\;\; c_1\;\; d)$$
$$Q_2\,(a_2\;\; b_2\;\; c_2\;\; d)$$

in which $d_1 = d_2 = d$. Here are some examples.

5^2 – *kolam*. (Figure 14A). The quartets are

$$Q_1\,(1\;\; 2\;\; 3\;\; 5)$$
$$Q_2\,(3\;\; 1\;\; 4\;\; 5)$$

At the centre is a linear 1 x 3 string. All the six *kolam*s (a) – (f) are based on the same pair of quartets, each with a different 2 x 4 rectangle, but all with the same number of splices 8. This is maximal in (c) and (d); in others an extra pair is allowed but leads to multiple loops.

10^2 – *kolam*. (Figure 14B). The quartets are

$$Q_1\,(4\;\; 3\;\; 7\;\; 10)$$
$$Q_2\,(6\;\; 2\;\; 8\;\; 10)$$

The central 4 x 6 rectangle is the same as Figure 8(f). The 3 x 8 rectangle is from Figure 3. The number of splices is 2 x 8 = 16.

11^2 – *kolam*. (Figure 14C). The quartets are

$$Q_1\,(3\;\; 4\;\; 7\;\; 11)$$
$$Q_2\,(5\;\; 3\;\; 8\;\; 11)$$

The central rectangle 3 x 5 is from Figure 9(g). The rectangle 4 x 8 is from Figure 10A and the 7 x 3 is from Figure 12A.

9^2-*kolam*. (Figure 14D). In this *kolam* the central rectangle is also a square 3^2. This means the quartets are identical.

$$Q_1\,(3\;\; 3\;\; 6\;\; 9)$$
$$Q_1\,(3\;\; 3\;\; 6\;\; 9)$$

The rectangles 3 x 6 and 6 x 3 are chosen different so that the symmetry is only two-fold and not four-fold. A casual glance shows they are different: the clover-leaf in 6 x 3 does not appear in the 3 x 6.

15^2 – *kolam*. (Figure 14E). Like the 9^2 *kolam* this too has a central square 5^2. The quartets are identical.

Q_1 (5 5 10 15)

Q_1 (5 5 10 15)

The rectangles 5 x 10 and 10 x 5 are chosen different. They are obtained by splicing two 5^2 *kolam*s side by side. The 15^2 *kolam* is therefore composed of nine 5^2 *kolam*s. In Figure 14A are given six different 5^2 *kolam*s (a) – (f). Of these (a) – (e) are positioned as shown. Note that the

(a) (b) (c)

(d) (e) (d)

(c) (b) (a)

composite *kolam* has two-fold rotational symmetry. The number of splices is 2 x 10 = 20.

8. Discussion and Summary.

Using a quartet of four integers *Q* (*a b c d*) based on Fibonacci Recurrence, square *kolam*s with four-fold rotational symmetry can be created (Figure 1) [1][2]. In this article rectangular *kolam*s with two-fold rotational symmetry are obtained with a pair of quartets

$$Q_1\ (a_1\ \ b_1\ \ c_1\ \ d_1)$$

$$Q_2\ (a_2\ \ b_2\ \ c_2\ \ d_2)$$

Here $c_1 = a_1 + b_1$, $d_1 = c_1 + b_1$, $c_2 = a_2 + b_2$ and $d_2 = c_2 + b_2$. Rectangular *kolam*s $d_1 \times d_2$ of any desired dimensions are possible. A few restrictions apply. The pair a_1, d_1 must be of same parity. Similarly the pair a_2, d_2 too must have the same parity (Section 2, Figure 2).

The key feature is the modular construction, building larger *kolam*s using smaller constituents (Section 3). I have presented 25 *kolam*s of varying sizes – from 2 x 3 to 15^2 - often using the smaller *kolam*s to build larger ones. All of them have the same basic construction. Five rectangles are joined together to make the desired *kolam*. This is done by splicing together two rectangles at points along the edges of the rectangles (Figure 2). All the points along the edges are potential splicing points and the final outcome of splicing at a set of points is one or more loops. Splicing pattern determines the number of loops (Section 4).

A single loop is harder to achieve. The simplest way to get a single loop is to splice each of the four rectangles to the central rectangle at a *single point.* A single loop results from merely four splices. More splices will enhance the richness and complexity of the design. For greater aesthetic quality, I have chosen two guiding principles: (1) a single loop (2) a set of splices that avoids closed four-sided islands (Section 4). The second rule generally limits the splices to alternate points along the edges of the rectangles. The two basic facts about splicing are

(a) a splice *between* two loops gives one loop and

(b) a splice *within* a single loop splits it into two loops.

This implies that a splice always results in an increase or decrease of the number of loops by one. As the number of splices increases with the size of the *kolam*, the number of loops tends to small limiting values. Starting with one loop even after 20 splices the number of loops is one or three. Since splices are done in symmetrically placed pairs, the parity of the number of loops is preserved – in the present case 'odd'. An optimal strategy is to aim for maximum number of splices consistent with a single loop. This may involve tweaking the number of splices by removing (unsplicing) some splices (Section 5).

A simple and elegant variant of the rectangular *kolam* is a 'Window-frame' *kolam* with the central rectangle omitted (Section 6). In Figure 13A and Figure 13B are given two versions of 9 x 13 windows. They differ from all the other *kolams* in one very important respect: the number of loops is two. This is because the starting configuration had four loops (without the central rectangle) an even number. Parity conservation determines the minimum number of loops as two.

There is a simple way to estimate the maximum number of splicing points for a given rectangular *kolam*. In Figure 2 the splices occur along lines of length c_1 and c_2. Since only alternate points are spliced, the number of splices is $\approx (c_1 + c_2)/2$. But these splices are duplicated for symmetry. So the maximum number of splices is $\approx (c_1 + c_2)$. A small correction δc is to be added to $(c_1 + c_2)$ to account for extra splice for odd c_1 or c_2 (e.g. for $c_1 = 5$, there are three splicing points at positions 1,3,5) and occasional adjacent splicing points that are allowed. It turns out that δc can take values 0,1,2,3. In the table below one example for each value of δc is given;

d_1 x d_2	Figure	c_1	c_2	#splices	δc
7 x 19	11**B**	4	12	16	0
10^2	14B	7	8	16	1
4 x 8	10A	3	5	10	2
9 x 13	12B	7	10	20	3

In the 25 *kolam*s in this paper, there are 7, 6, 9 and 3 *kolam*s respectively with δc = 0,1,2,3. The mean $< \delta c > = 1.3$. The sum $(c_1 + c_2)$ can be written in terms of a's and d's

$$c_1 + c_2 = (a_1 + d_1)/2 + (a_2 + d_2)/2 = (a_1 + a_2)/2 + (d_1 + d_2)/2$$

It is one-half the sum of the sides of the outer and inner rectangles.

Square *kolam*s are special cases of rectangular *kolam*s. Here they have only two-fold rotational symmetry, unlike the square *kolam*s mentioned in the Introduction. The lesser symmetry leads to a wider choice of quartets and greater variety of *kolams*. (Section 7, Figures 14A – 14E).

Before concluding this section a comment is warranted about the splicing rule: *'a splice within a single loop splits it into two loops'*. It was mentioned in Section 4 that in principle a splice within a single loop can leave the loop single, but this is a rare occurrence. In seeking an explanation for this it was found that splicing points are of two types: the directions of traversal of loop on either side of the splice are *parallel* (↑↑) or *anti-parallel* (↑↓). For parallel splices the result is two loops and for anti-parallel, one loop. I have examined many completed *kolam*s with single loop, for any remaining allowed splicing points. Almost all of them are parallel yielding two loops. Further, when no additional splices are available (i.e. the *kolam* has already maximal splicing), tracking the loop around the other disallowed points, shows that the chances of parallel and anti-parallel splices are about equal. These 'illegal' splicing points are generally adjacent to the allowed splicing points. The connection between splices and loops is explored in a paper in preparation [6].

Although some rules are known about splices and loops, one needs to experiment with them to achieve maximum splicing consistent with a single loop. The effort can be time consuming, especially for larger *kolam*s. Clearly there is a need for aid of the computer. It is found that all the *kolam*s described can be dissected into eight basic types of units cells (the squares with a dot in the centre) [7]. This is a major step forward in the computer-aided design of *kolam*s. □

Acknowledgment

In the living room of the house I live, there is a large wall cupboard made up of four rectangles in two striking contrasting colours of red and white. The proportion of the sides is very similar to that of the Window-frame *kolam* (Section 6). This was the inspiration for this paper.

My daughter Venil Sumantran has watched closely the evolution of this paper and with useful suggestions and gentle prodding helped me finish this paper in a short time. I am thankful to her.

I learnt to draw my first *kolam* at age eight from my mother (Thailammal). Ever since, I have sustained my interest in the art and have explored the underlying mathematics. I owe my interest in mathematics to my father and teacher (K.N. Sundaresan), a professor of mathematics who was also a prolific poet in Tamil. This paper is dedicated to their memory.

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10B	7 x 22
10C	7 x 22 (Skeleton)
11A	4 x 7 and 3 x 12
11B	7 x 19
12A	2 x 10, 3 x 7 and 5 x 7.
12B	9 x 13
13A	9 x 13 (Window-frame I)
13B	9 x 13 (Window-frame II)
14A	5^2
14B	10^2
14C	11^2
14D	9^2
14E	15^2

Chennai, 11 November 2015 *S. Naranan*

KOLAM DESIGNS BASED

ON FIBONACCI NUMBERS

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Part III. Rectangular *Kolams* with

two-fold Rotational Symmetry

S. Naranan

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1. Introduction.
2. Rectangular *Kolam*s with two-fold Rotational Symmetry
3. Choice of Quartets
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5. Pi-*kolam* and e-*kolam*
6. Window-frame *kolam*
7. Square *Kolam*s
8. Discussion and Summary

Acknowledgment

References

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Figure captions.

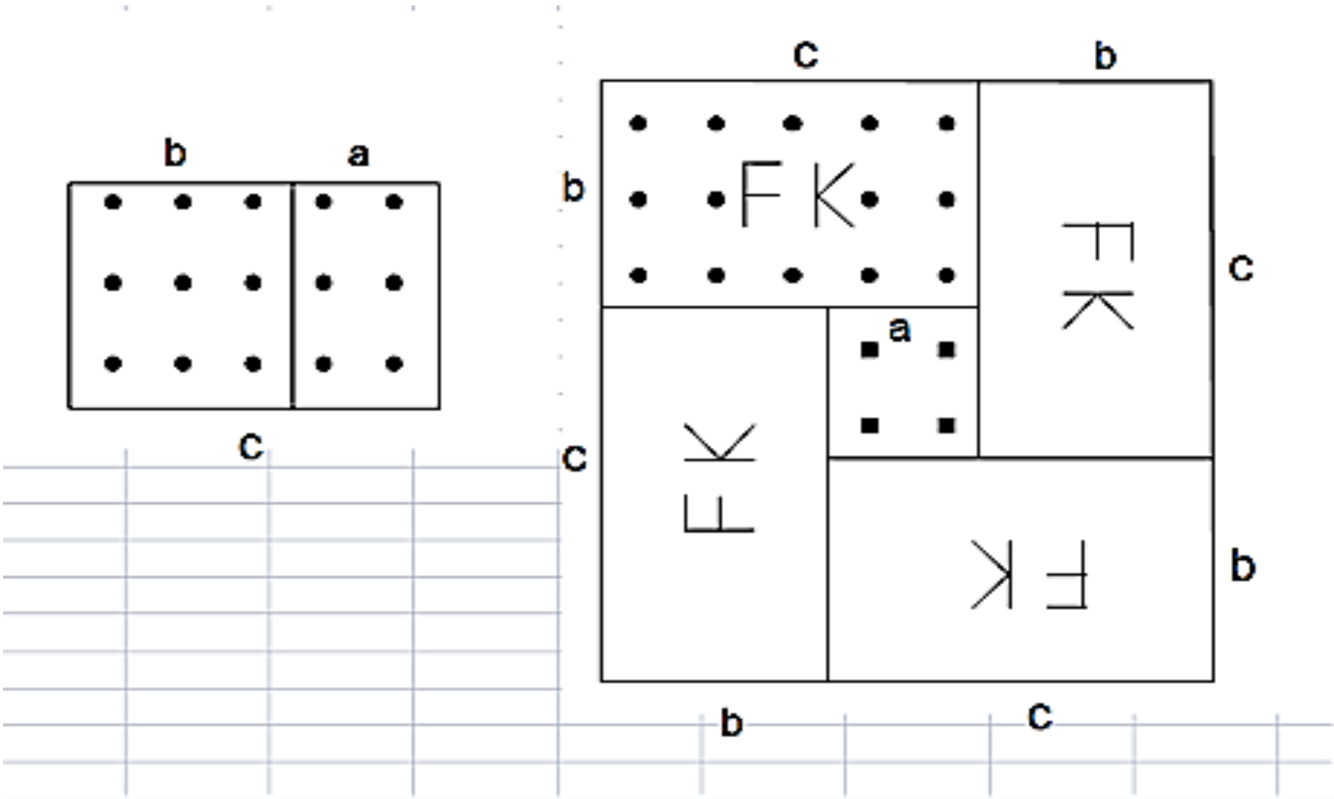

Figure 1

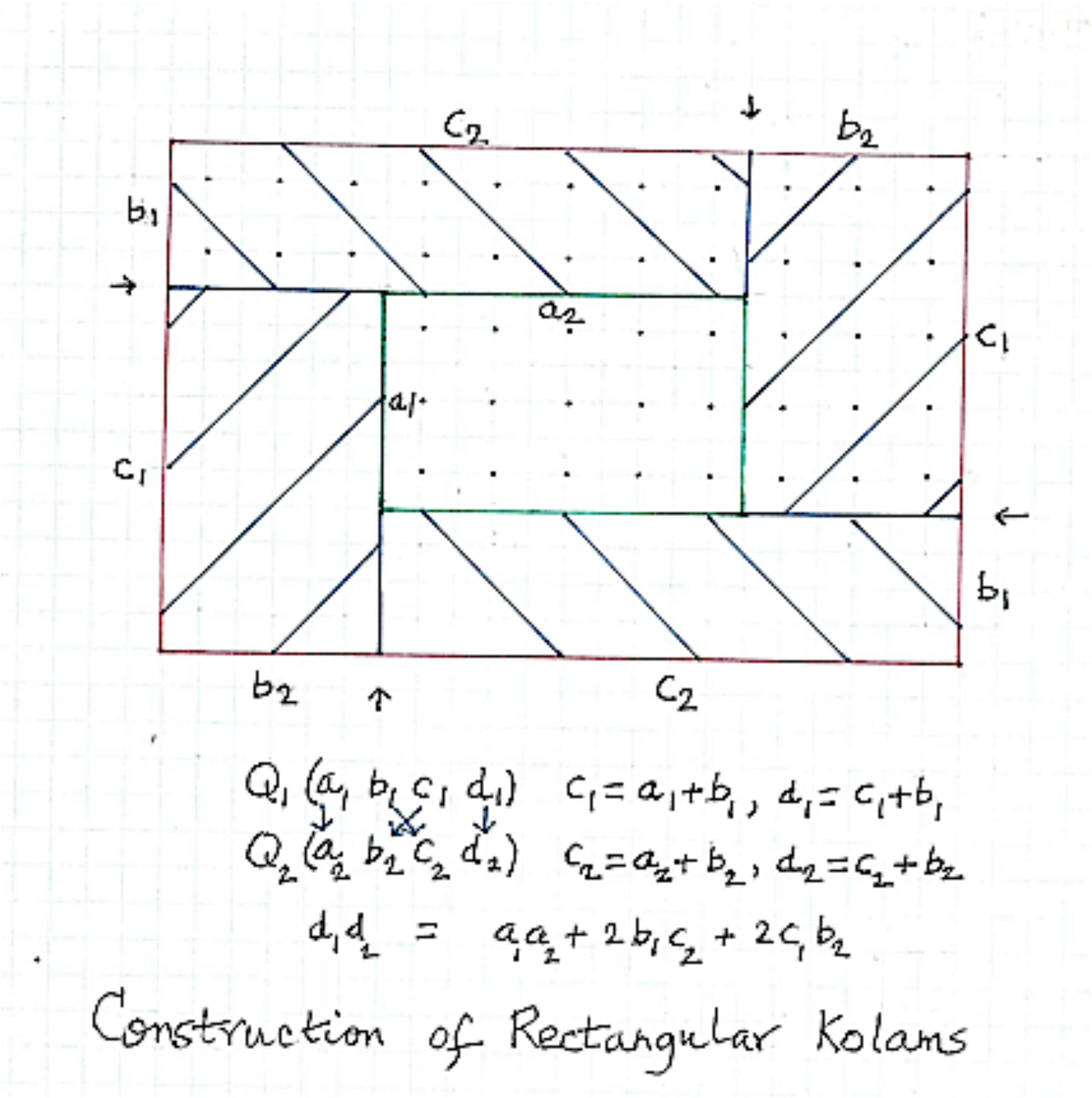
c_2
b_2
b_1
a_2
c_1
d_1
c_1
b_1
b_2
c_2
$Q_1\,(a_1\; b_1\; c_1\; d_1)$ $\quad c_1 = a_1 + b_1,\; d_1 = c_1 + b_1$
$Q_2\,(a_2\; b_2\; c_2\; d_2)$ $\quad c_2 = a_2 + b_2,\; d_2 = c_2 + b_2$
$d_1 d_2 = a_1 a_2 + 2 b_1 c_2 + 2 c_1 b_2$
Construction of Rectangular Kolams

Figure 2

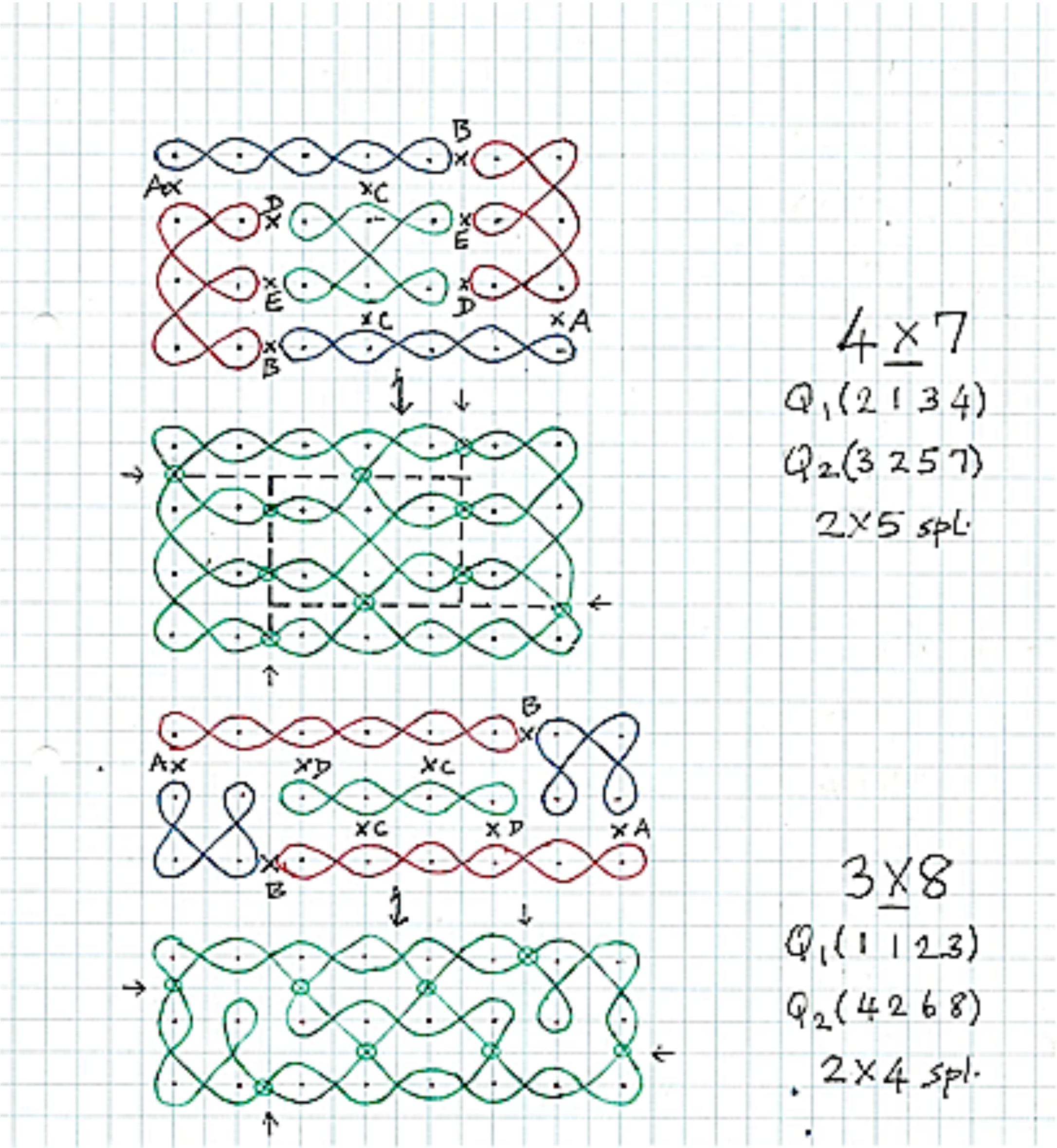

Figure 3

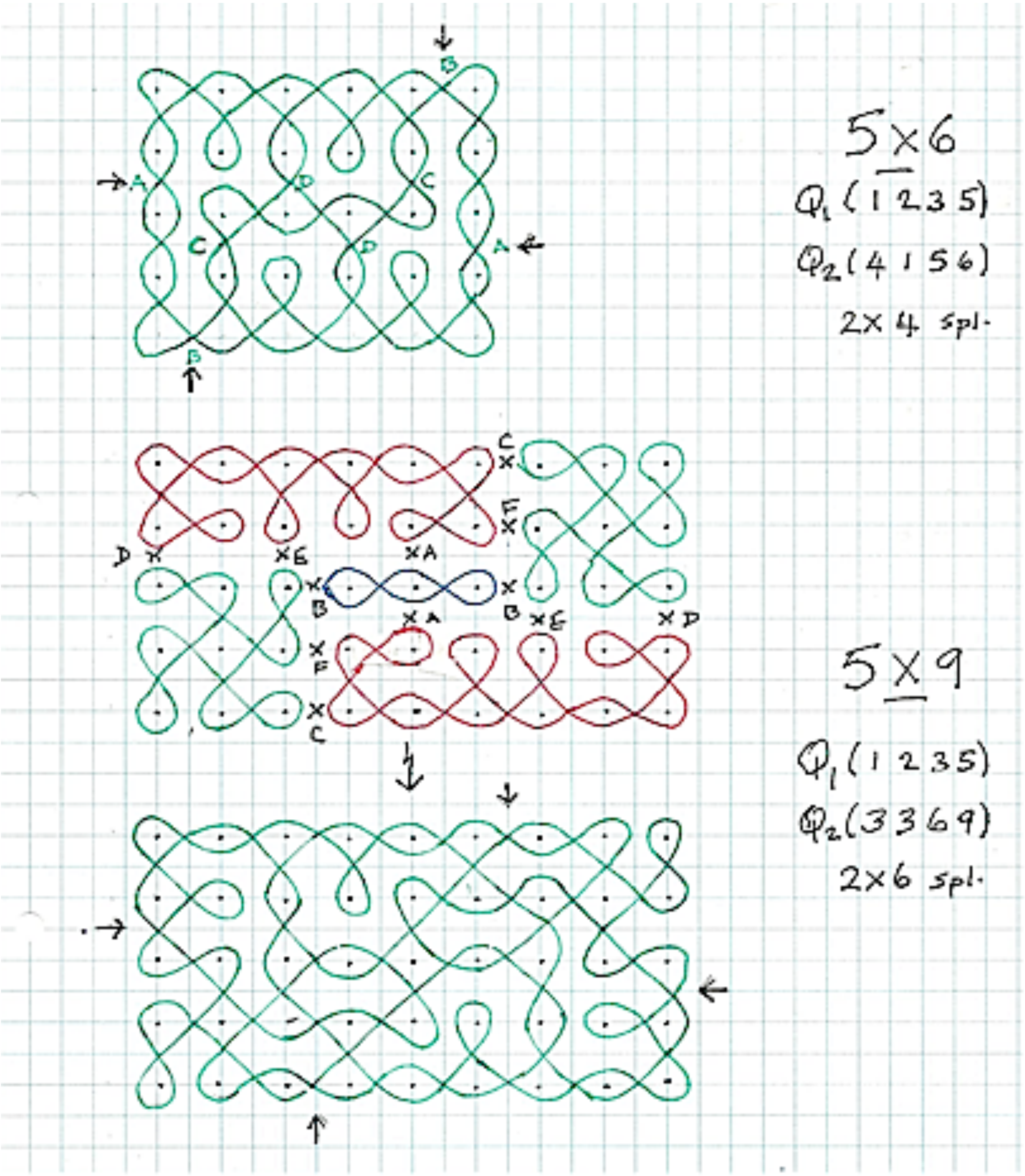

Figure 4

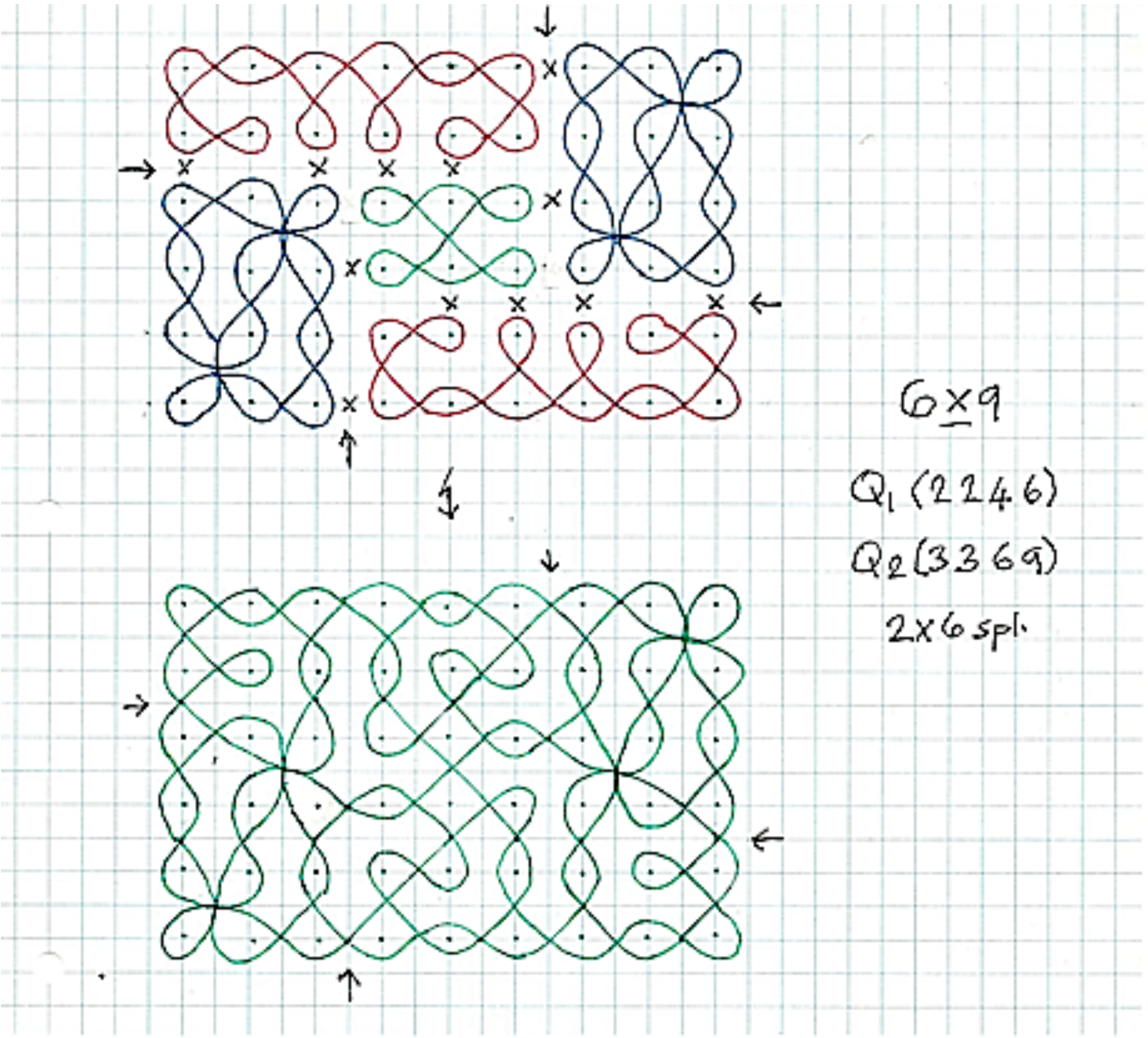

Figure 5

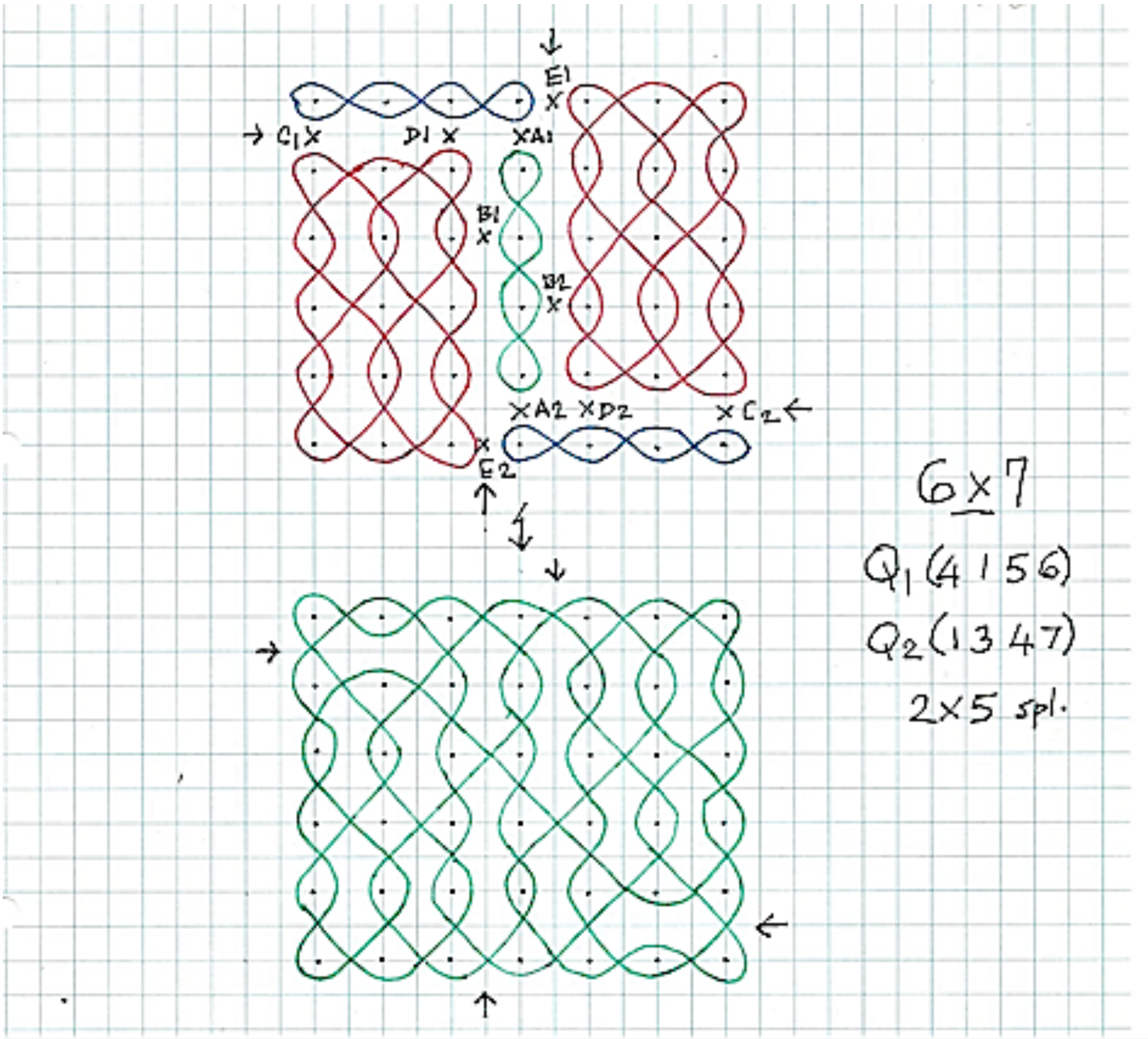

Figure 6

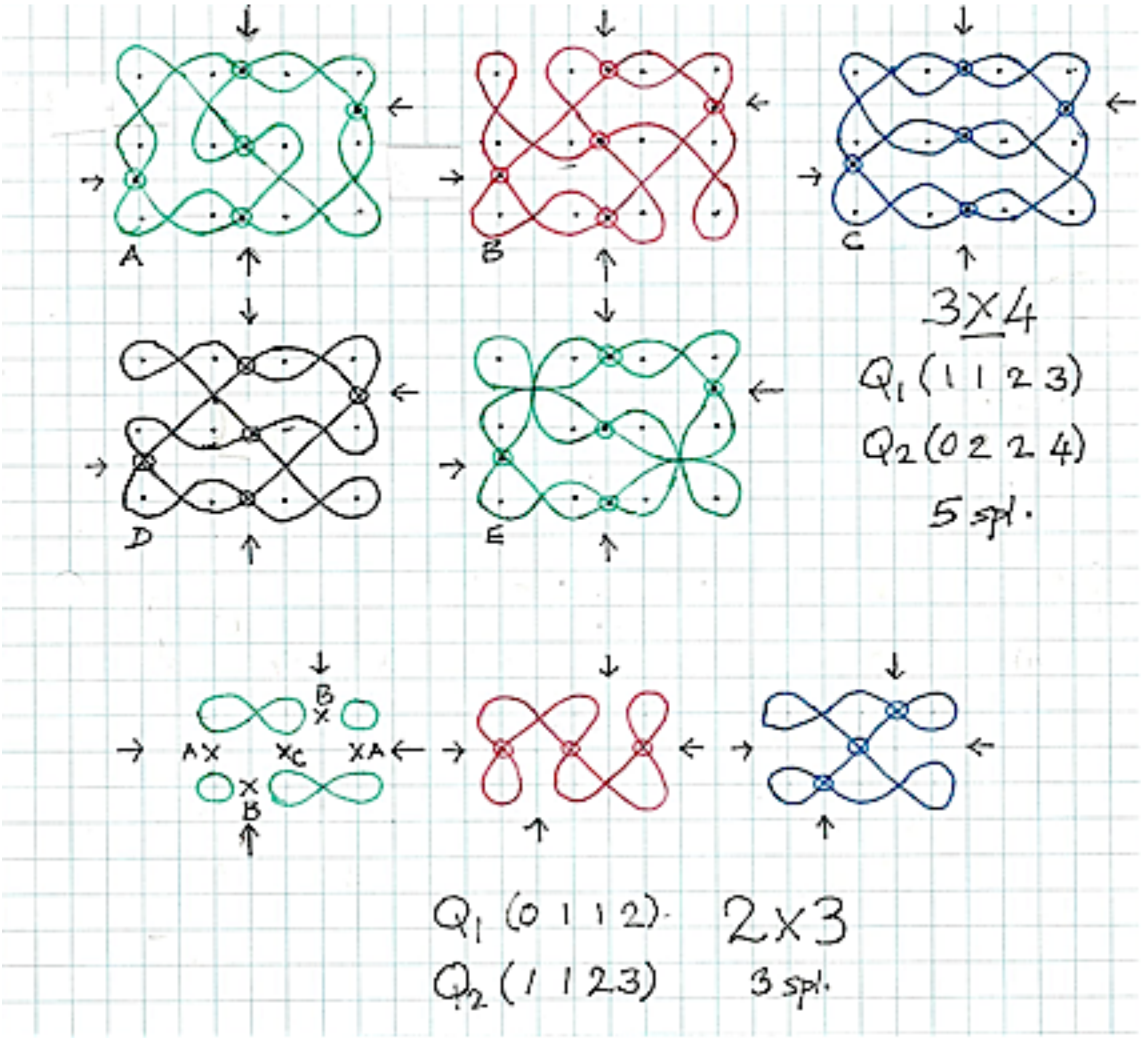

Figure 7

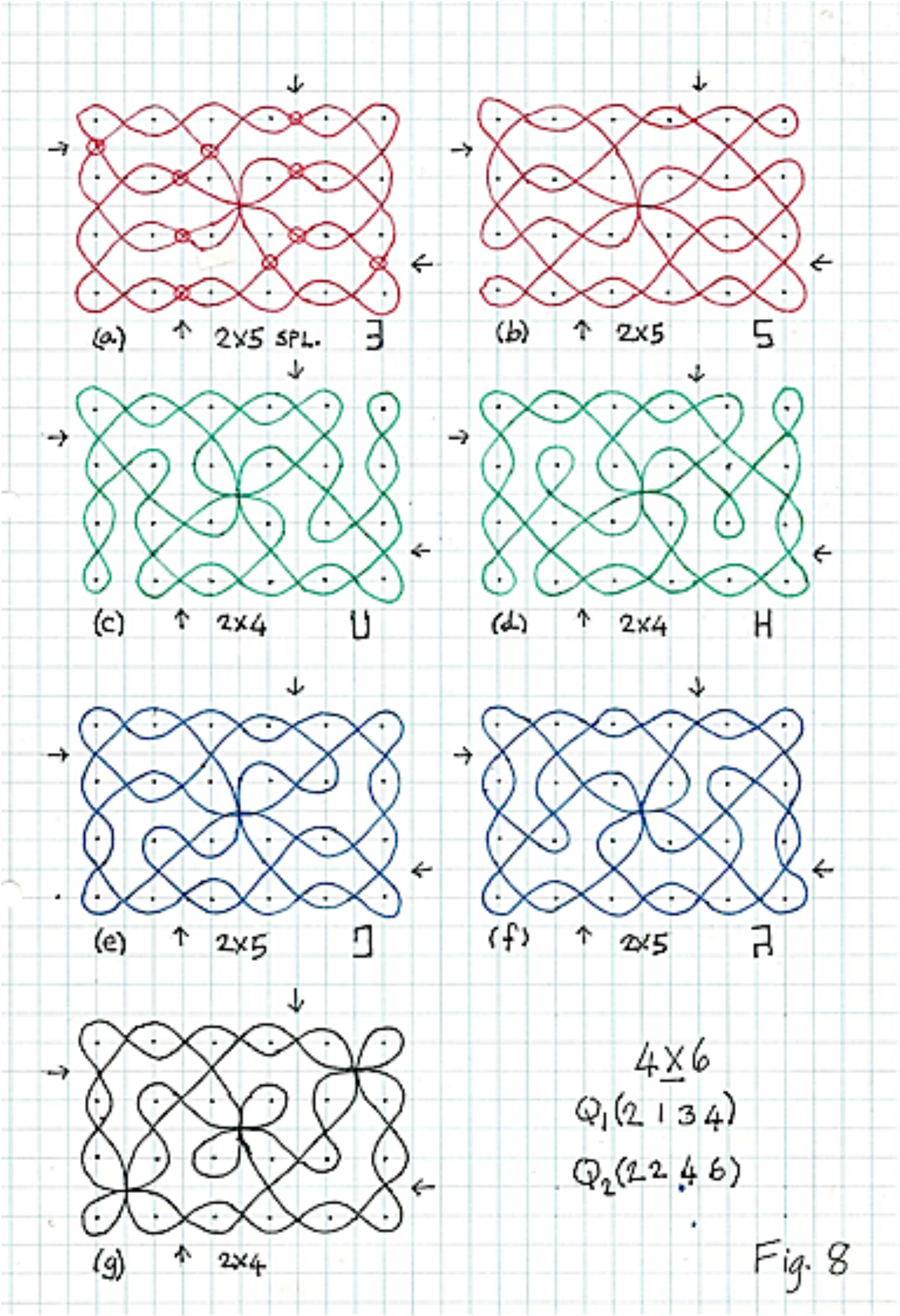
(a) 2x5 SPL.
(b) 2x5
(c) 2x4
(d) 2x4
(e) 2x5
(f) 2x5
(g) 2x4
4X6
Q1(2 1 3 4)
Q2(2 2 4 6)
Fig. 8

Figure 8

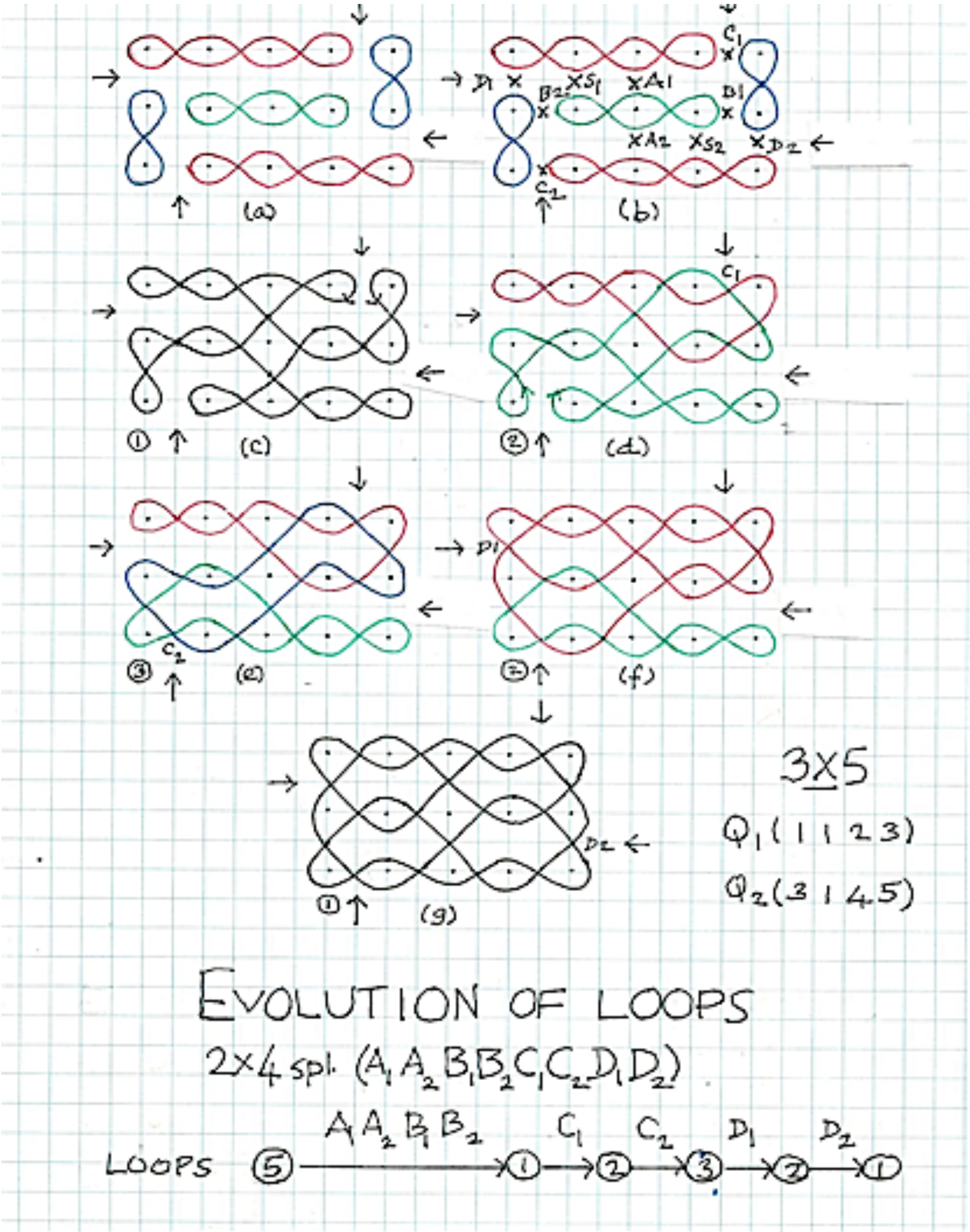
(a)
(b)
(c)
(d)
(e)
(f)
(g)
3X5
Q_1(1 1 2 3)
Q_2(3 1 4.5)
EVOLUTION OF LOOPS
2x4 spl. ($A_1 A_2 B_1 B_2 C_1 C_2 D_1 D_2$)
$A_1 A_2 B_1 B_2$ C_1 C_2 D_1 D_2
LOOPS 5 → 1 → 2 → 3 → 2 → 1

Figure 9

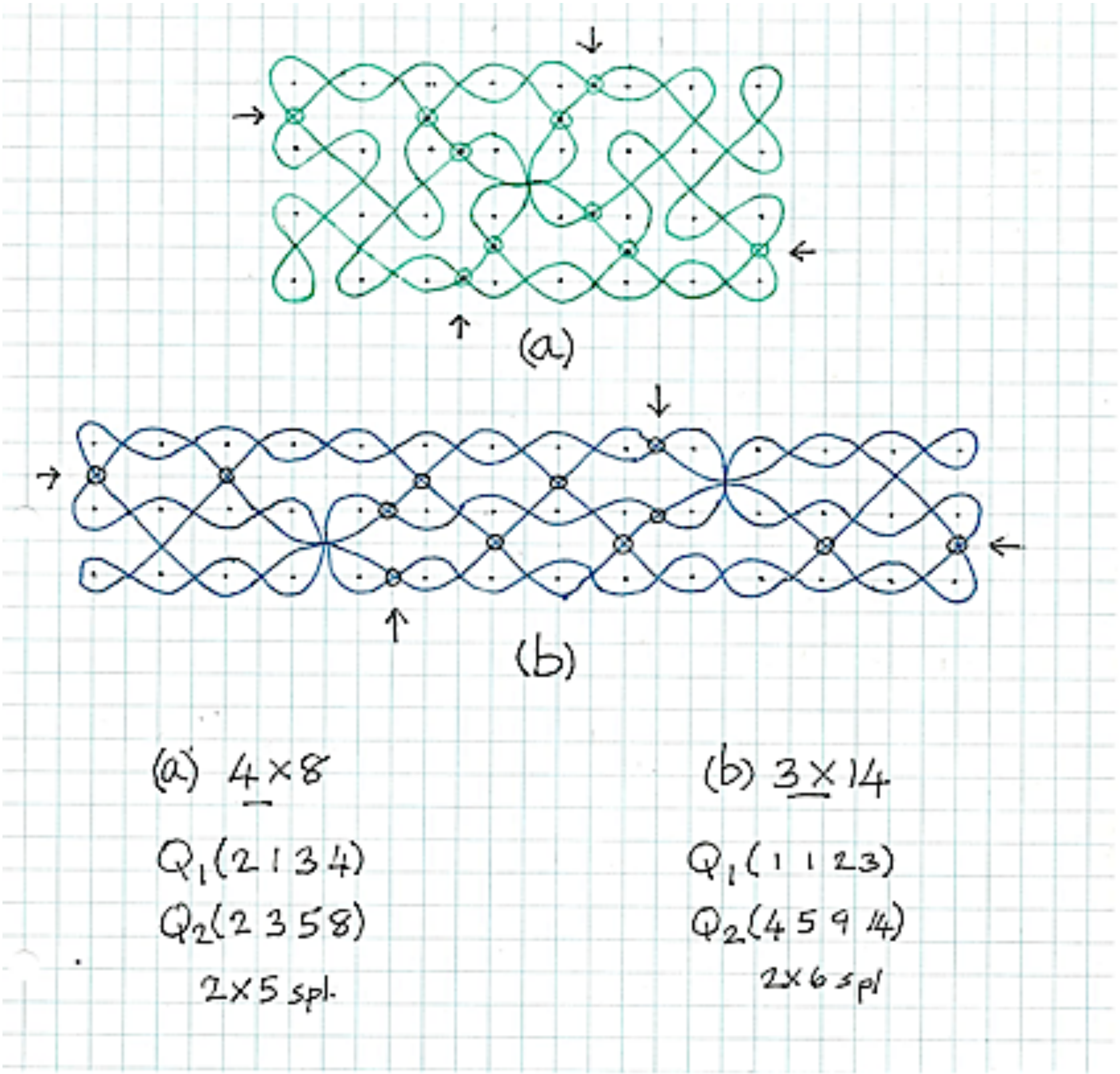

Figure 10 A

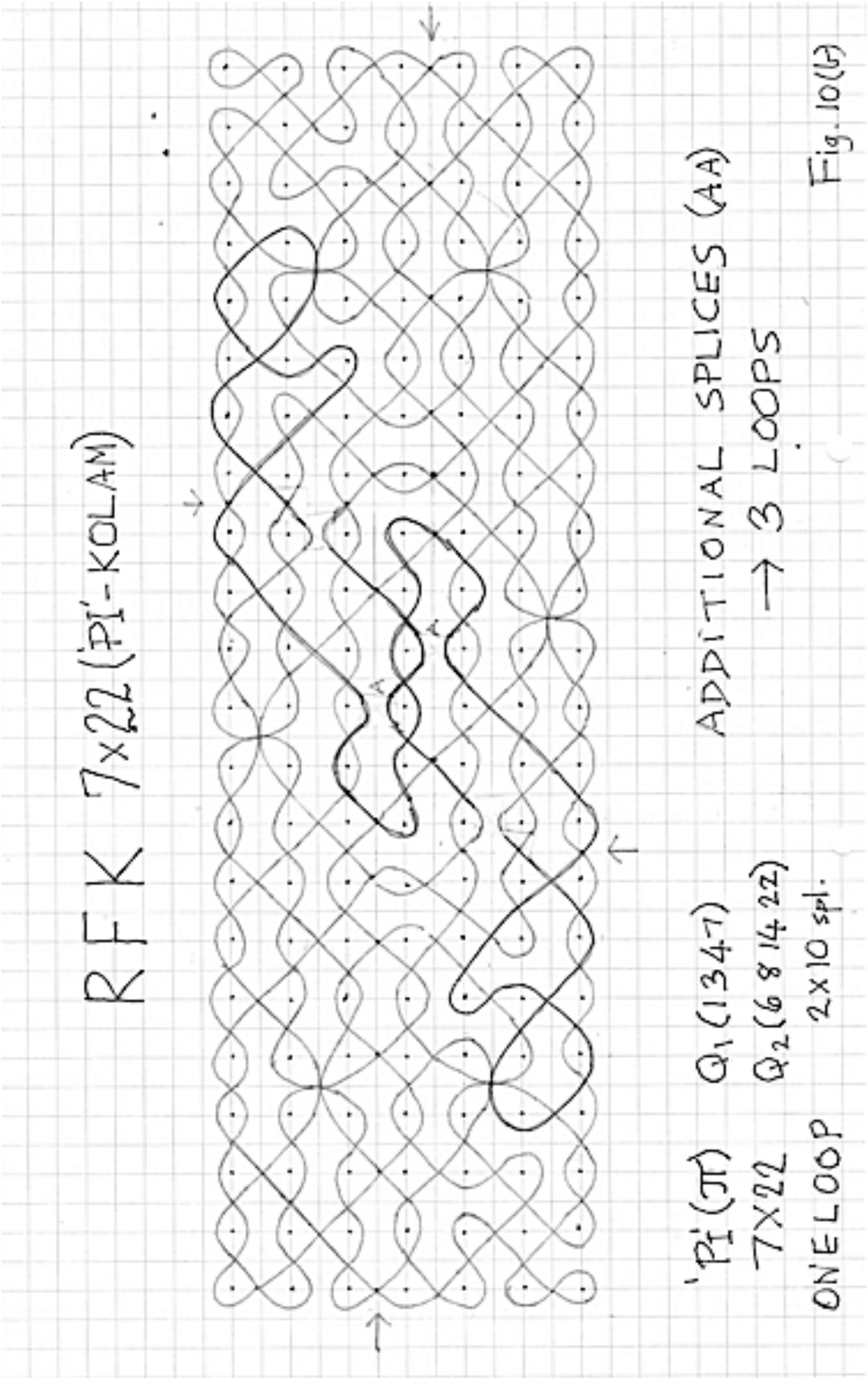
RFK 7x22 ('PI'-KOLAM)
ADDITIONAL SPLICES (AA)
→ 3 LOOPS
Fig. 10(B)
'PI' (π)
7X22
ONE LOOP
Q1 (1347)
Q2 (6 8 14 22)
2X10 spl.

Figure 10 B

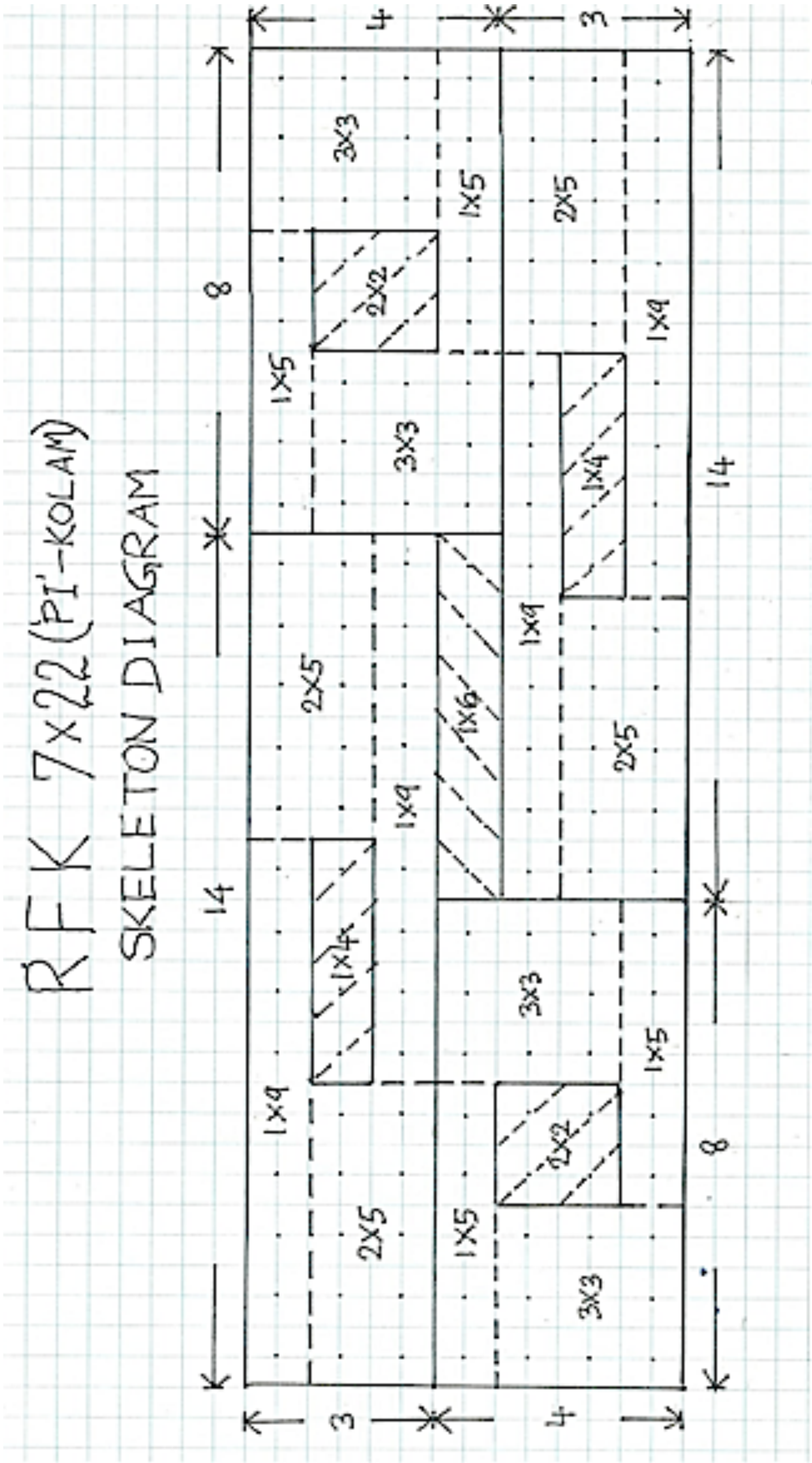
RFK 7X22 (PI'-KOLAM)
SKELETON DIAGRAM
8
14
14
8
4
3
3
4
3X3
1X5
2X5
2X2
1X9
1X4
3X3
2X5
1X6
1X9
1X4
2X5
1X9
3X3
1X5
2X2
2X5
1X5
3X3

Figure 10 C

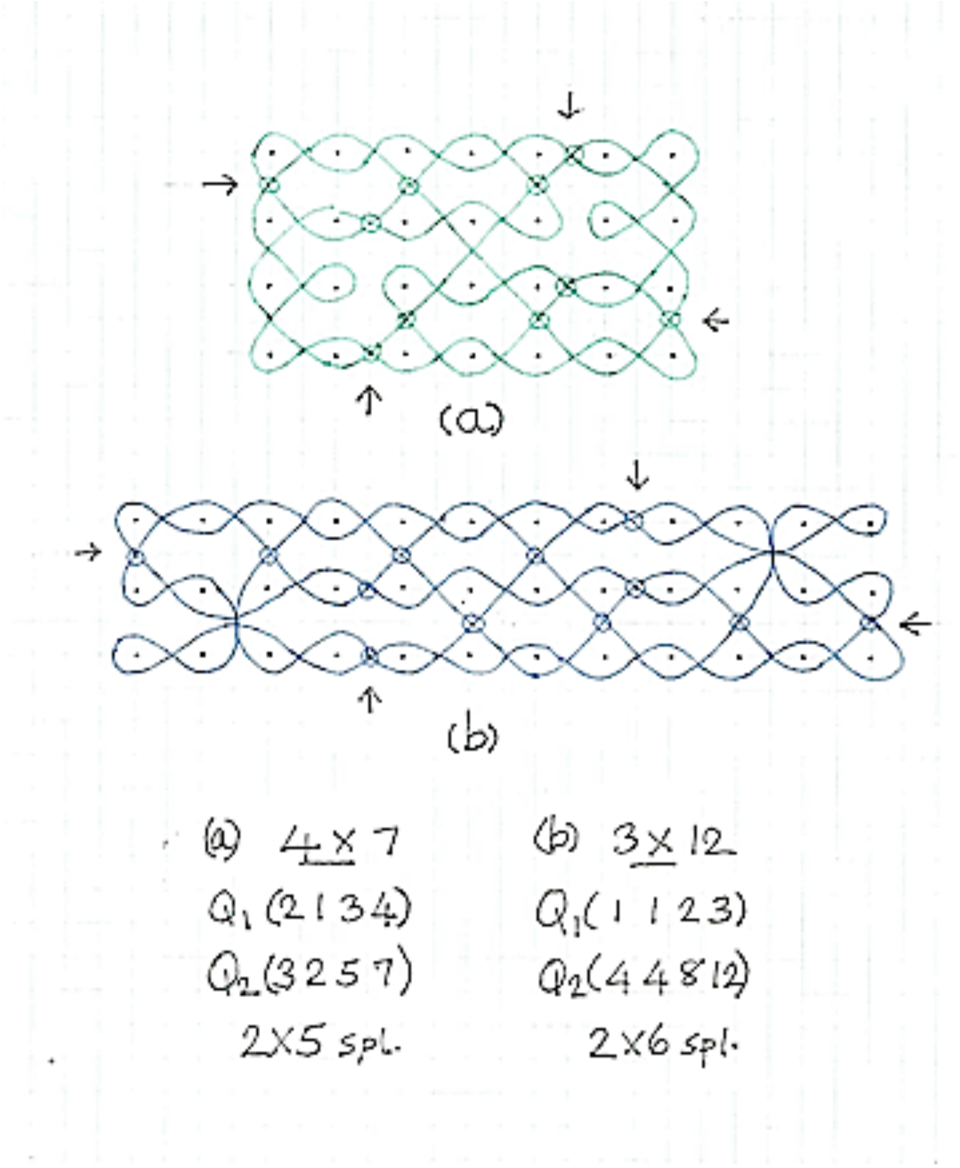

Figure 11 A

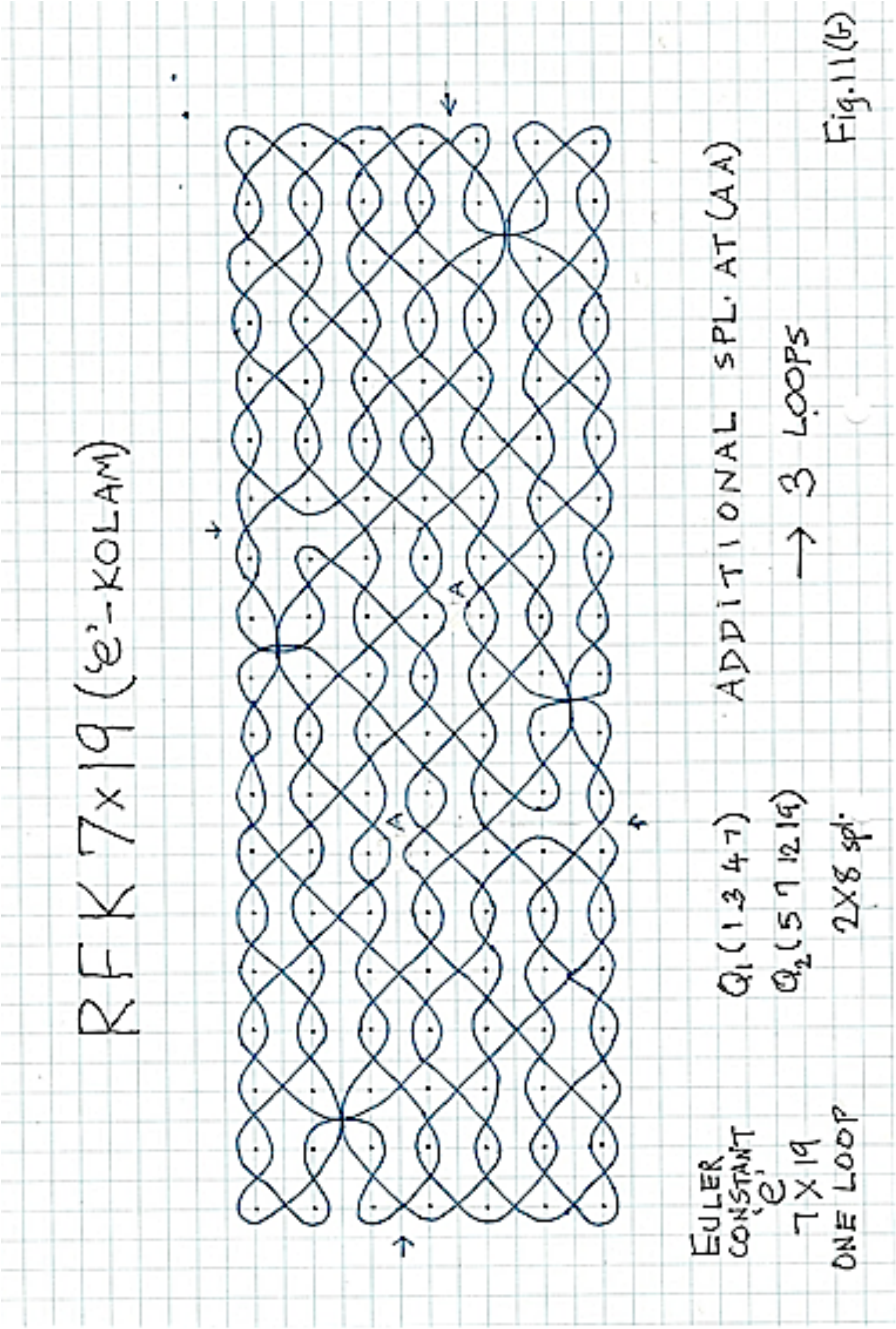
RFK 7x19 ('e'-KOLAM)
ADDITIONAL SPL. AT (AA)
Q1 (1 3 4 7)
Q2 (5 7 12 19)
2X8 spl.
→ 3 LOOPS
EULER CONSTANT 'e'
7 X 19
ONE LOOP
Fig.11(G)

Figure 11 B

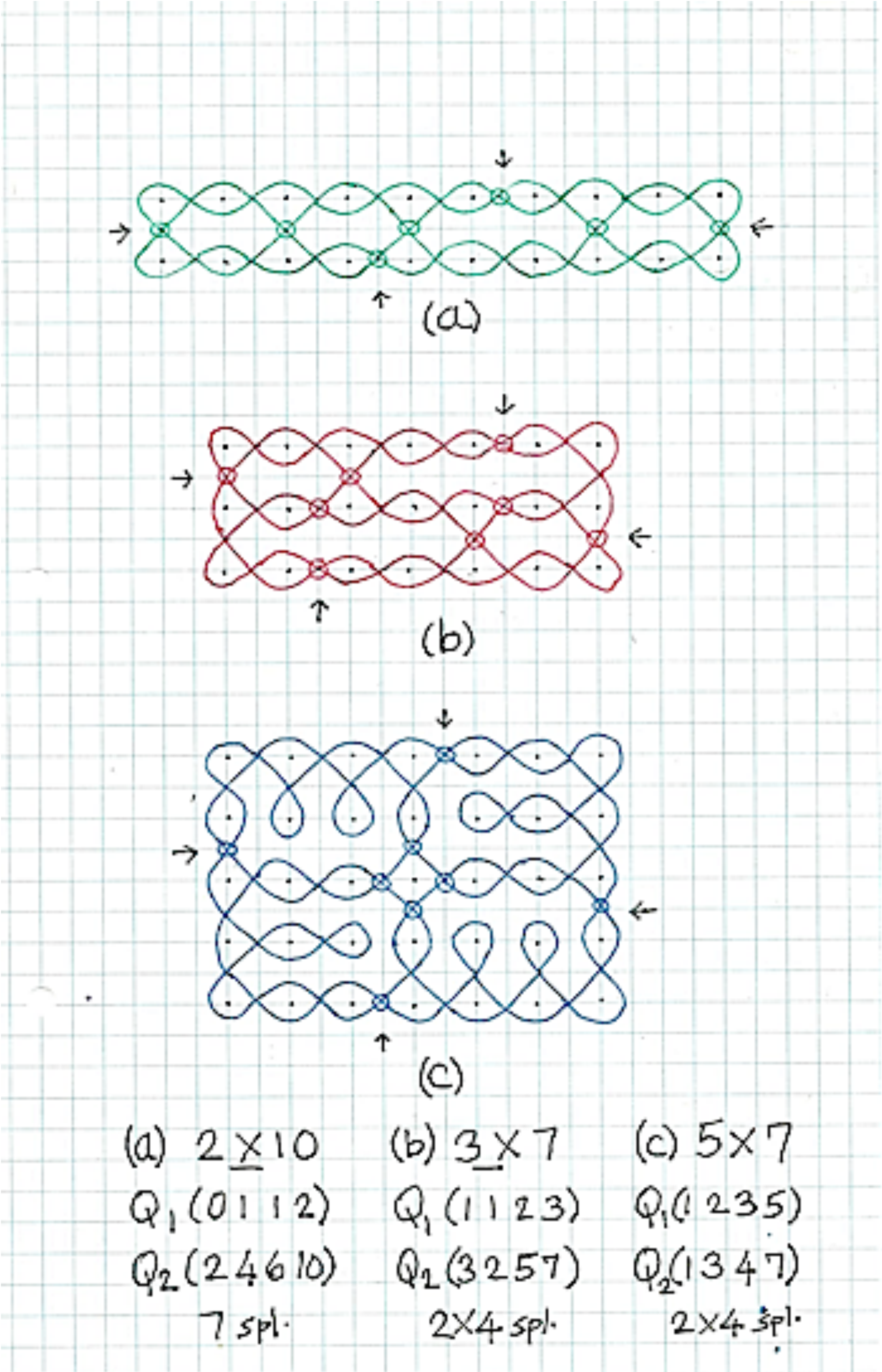
(a)
(b)
(c)
(a) 2×10
Q1 (0 1 1 2)
Q2 (2 4 6 10)
7 spl.
(b) 3×7
Q1 (1 1 2 3)
Q2 (3 2 5 7)
2×4 spl.
(c) 5×7
Q1 (1 2 3 5)
Q2 (1 3 4 7)
2×4 spl.

Figure 12 A

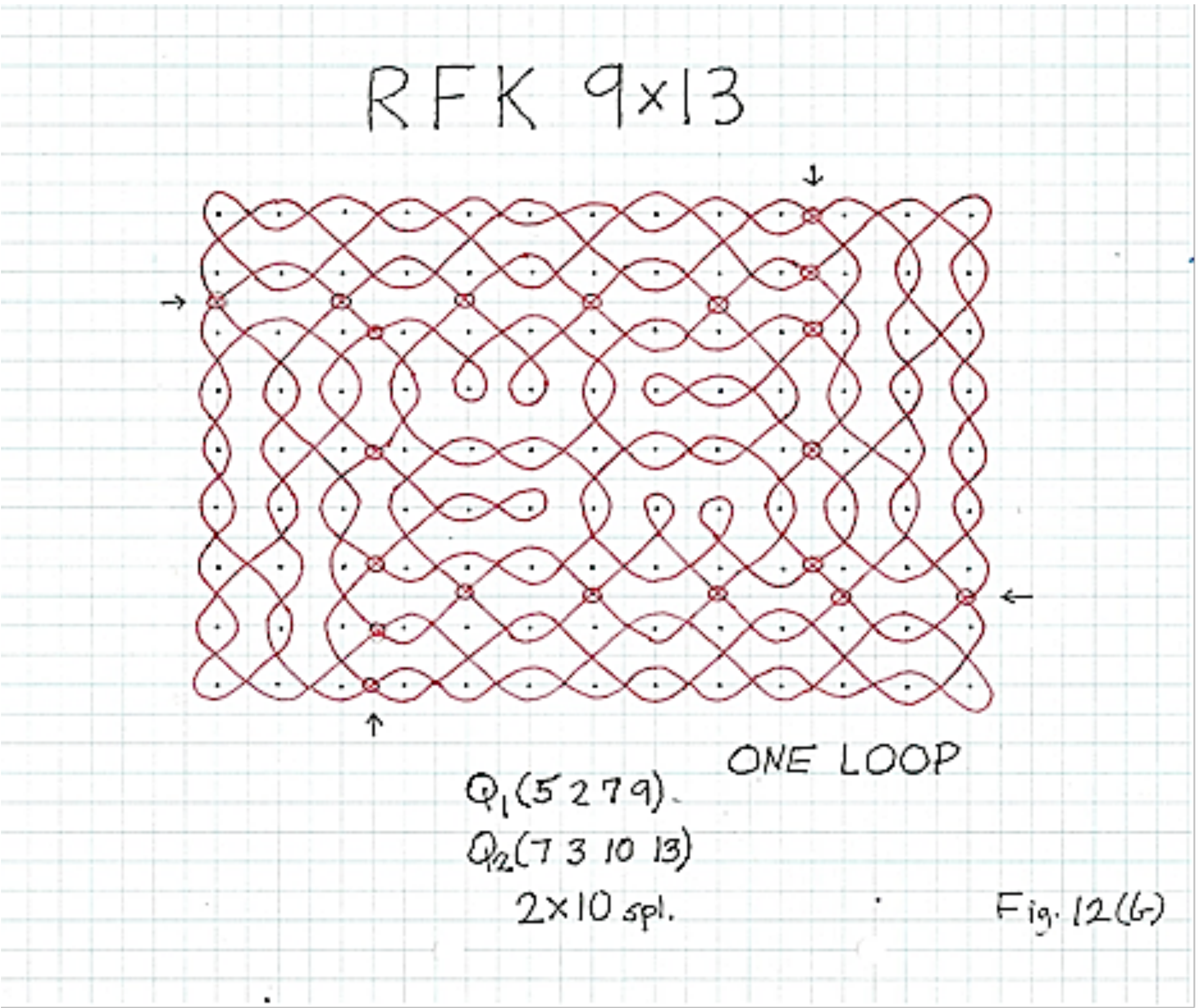

Figure 12 B

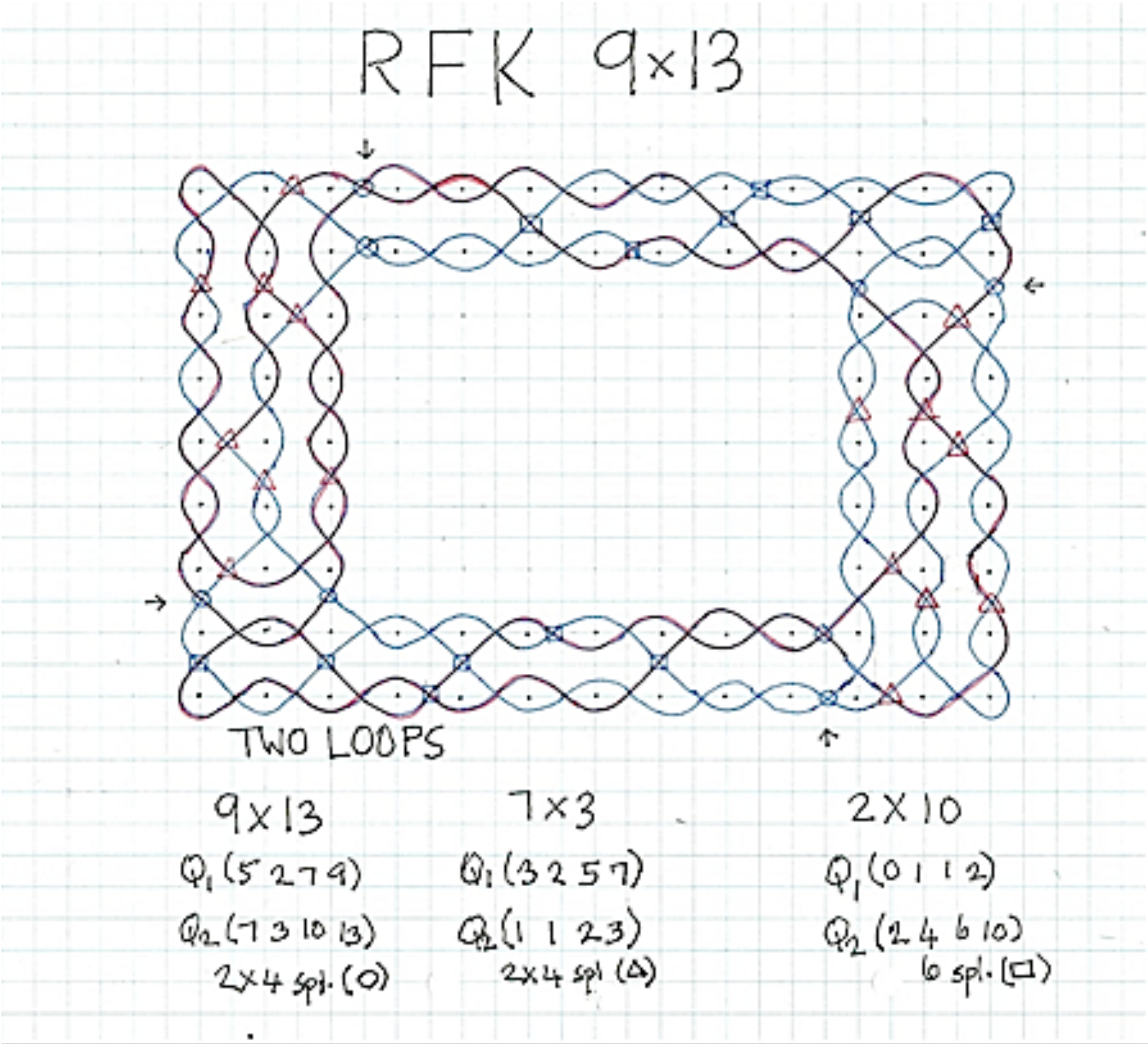
RFK 9×13
TWO LOOPS
9×13
Q1 (5 2 7 9)
Q2 (7 3 10 13)
2×4 spl. (O)
7×3
Q1 (3 2 5 7)
Q2 (1 1 2 3)
2×4 spl (△)
2×10
Q1 (0 1 1 2)
Q2 (2 4 6 10)
6 spl. (□)

Figure 13 A

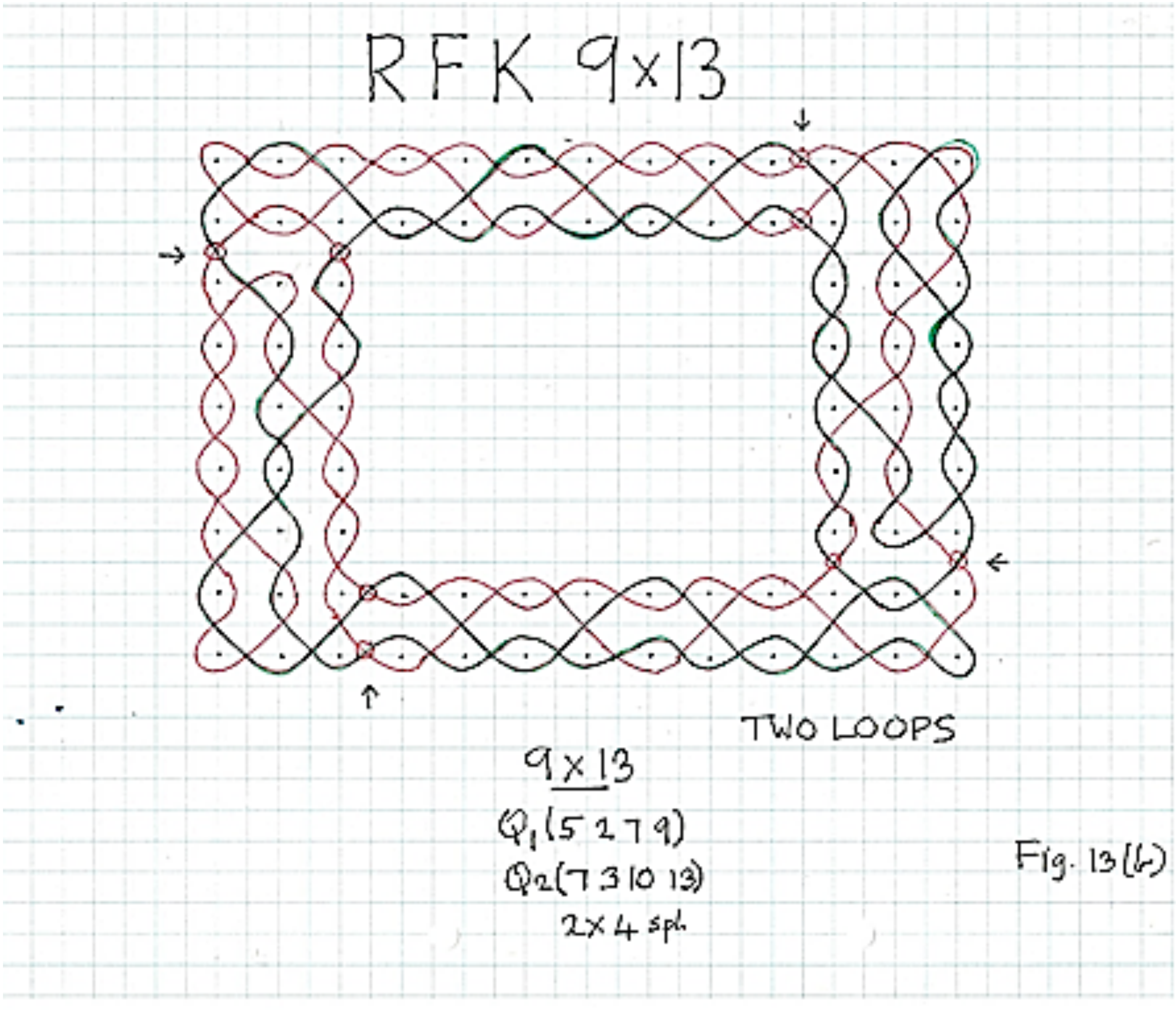

Figure 13 B

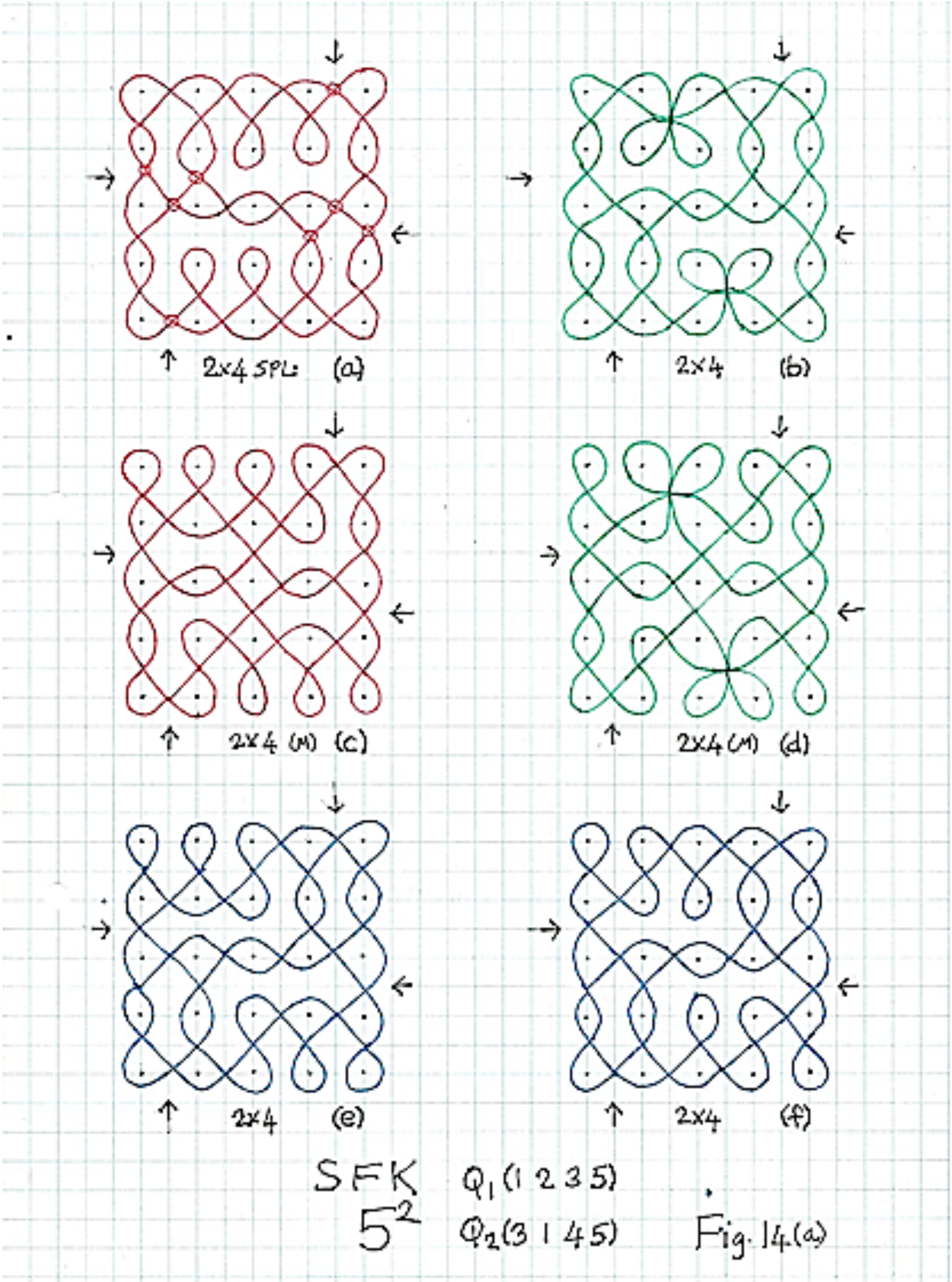
2x4 SPL: (a)
2x4 (b)
2x4 (M) (c)
2x4 (M) (d)
2x4 (e)
2x4 (f)
SFK
Q_1 (1 2 3 5)
5^2
Q_2 (3 1 4 5)
Fig. 14.(a)

Figure 14 A

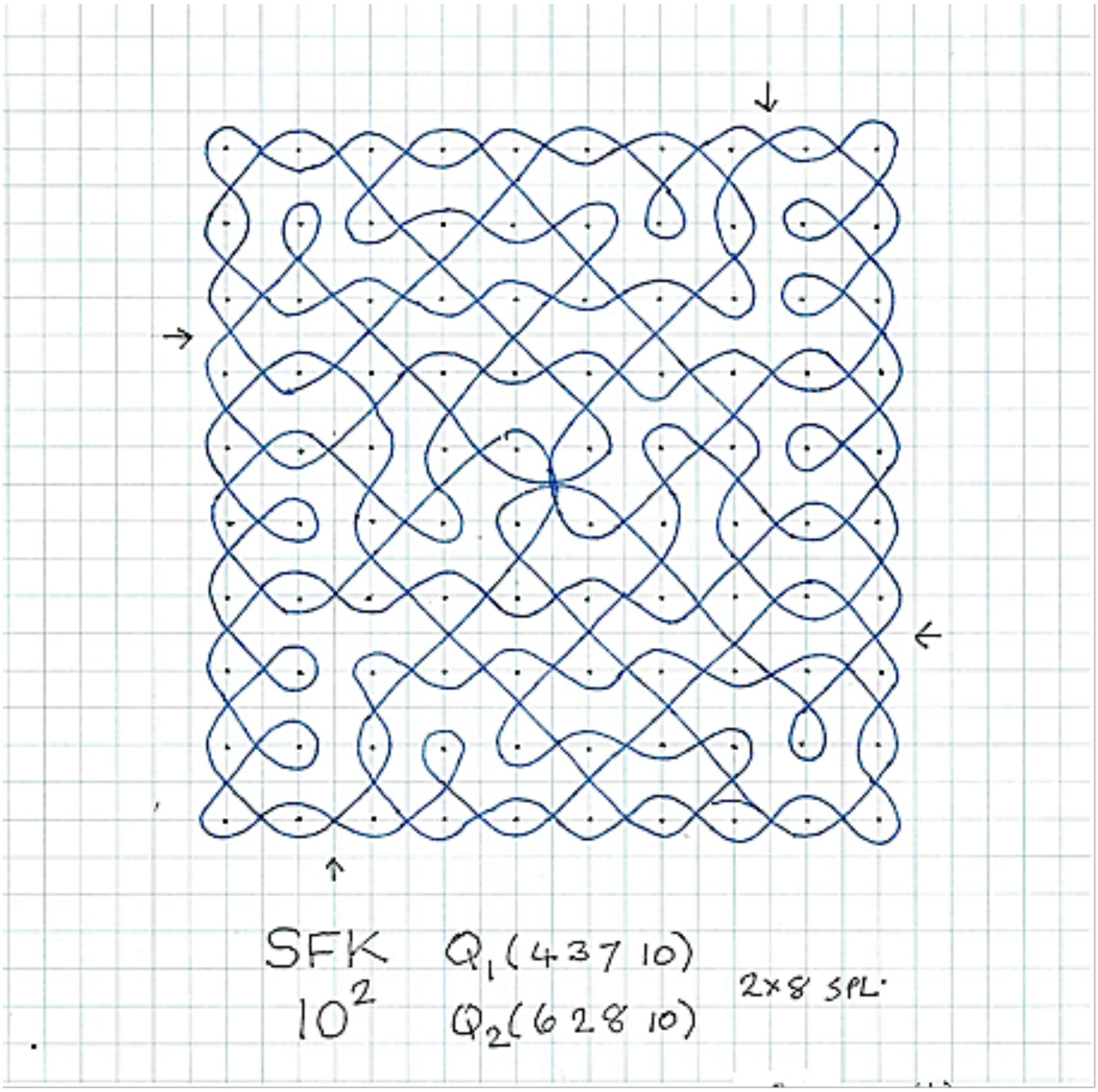

Figure 14 B

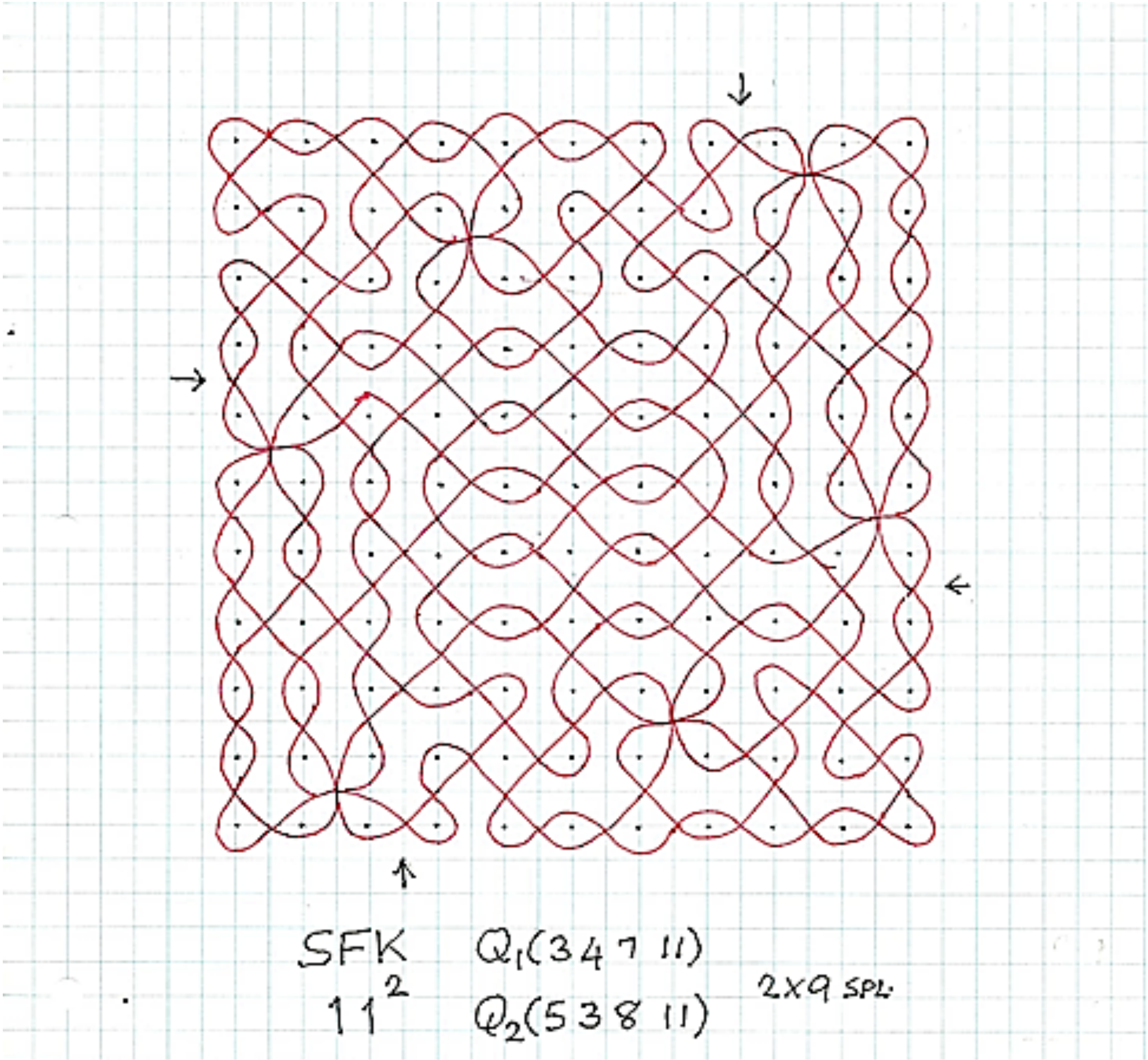

Figure 14 C

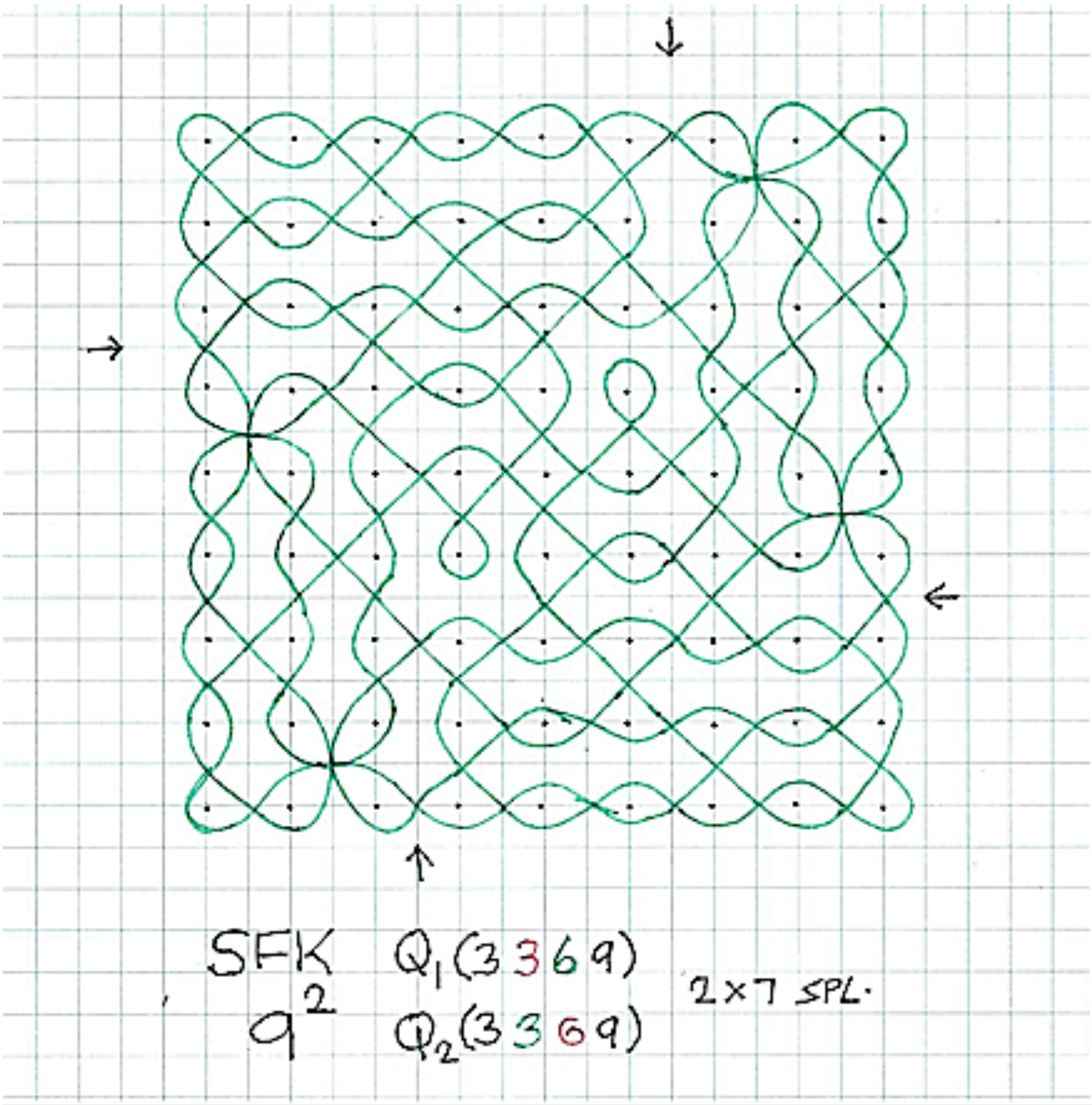

Figure 14 D

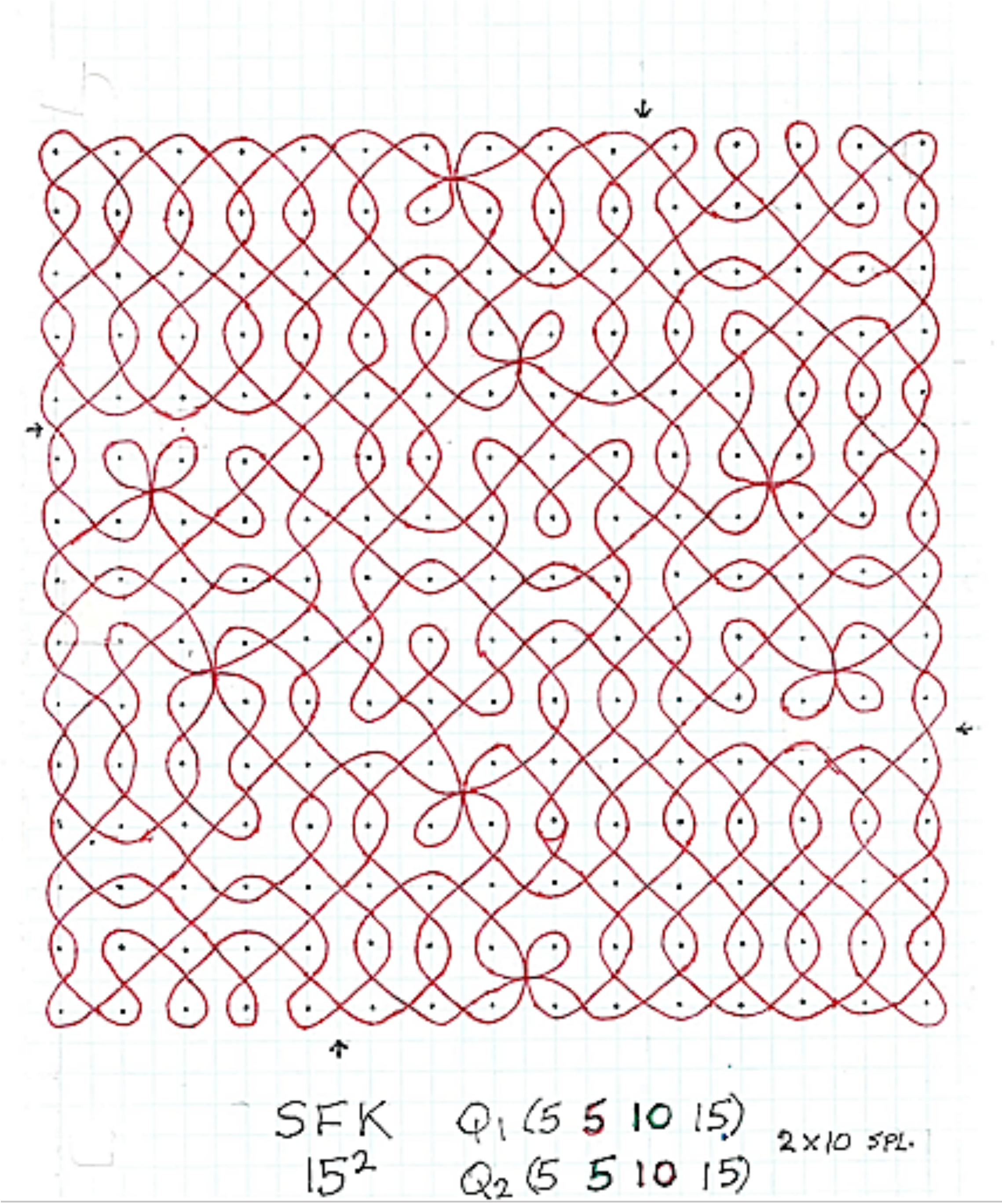

Figure 14 E